proprep™

Workbook

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3D Space

The 3D Coordinates System

Questions:

1) Find the projection of the point $P(4,7,-5)$ onto the three coordinate planes.

2) Answer the following questions:

a. What is the distance of the point $P(4,7,-5)$ from the z-axis?

b. Which point is closer to the z-axis: $P(4,7,-5)$ or $Q(5,-6,7)$?

3) Answer the following questions:

a. What is the distance of the point $P(4,7,-5)$ from the xy-plane?

b. Which point is closer to the xy-plane: $P(4,7,-5)$ or $Q(5,-6,7)$?

4) Answer the following questions:

a. Given the equation $2x+3y=6$,

i. In R^2 this is the equation of a:________

ii. In R^3 this is the equation of a:________

b. Given the equation $(x-1)^2+y^2=9$,

i. In R^2 this is the equation of a: ________

ii. In R^3 this is the equation of a: ________

Equations of Lines

Questions:

5) Find the three forms [vector, parametric, symmetric] of the equation of the line which passes through the points $(-10,4,0)$ and $(1,-4,2)$.

6) Find the three forms [vector, parametric, symmetric] of the equation of the line which passes through the point $(-10,4,0)$ and is parallel to the line $x=3+4t$, $y=-2+3t$, $z=-5t$.

7) Let l_1 be the line through points $(4,1,-5)$ and $(2,0,9)$ and let l_2 be the line given by $\vec{r}(t)=\langle 5,1-9t,-8-4t\rangle$. Are the lines l_1 and l_2 parallel, perpendicular or neither?

8) Let l_1 be the line given by $x=-7+12t$, $y=3-t$, $z=14+8t$ and let l_2 be the line given by $\vec{r}(t)=\langle 8+t,5+6t,4-2t\rangle$. Do l_1 and l_2 intersect? If so, find the intersection point.

9) Let l_1 be the line passing through points $(-5,0,2)$ and $(13,-2,1)$ and let l_2 be the line given by $\vec{r}(t)=\langle 3,-1-t,2+4t\rangle$. Do l_1 and l_2 intersect? If so, find the intersection point.

10) Let l be the line given by $x=-7+12t$, $y=3$, $z=16+8t$.

a. Does l intersect the xy-plane? If so, where?
b. Does l intersect the xz-plane? If so, where?

Equations of Planes

Questions:

11) Find the equation of the plane through the points $\mathrm{P}(0,1,1)$, $\mathrm{Q}(1,0,1)$ and $\mathrm{R}(1,-3,-1)$.

12) Find the equation of the plane passing through the point $(0,2,-1)$ and orthogonal to the line $\vec{r}(t)=\langle 5+t,1+3t,4t\rangle$.

13) Find the equation of the plane containing the point $(-7,3,9)$ and parallel to the plane $4x+8y-2z=37$.

14) Plane π_1 is given by $4x+8y-2z=10$ and plane π_2 is given by $2x+y+8z=11$. Are the planes π_1 and π_2 parallel, orthogonal or neither?

15) Plane π_1 is given by $2x-3y+4z=5$ and plane π_2 passes though points $(1,2,2)$, $(2,2,3)$ and $(-3,-2,-6)$. Are the planes π_1 and π_2 parallel, orthogonal or neither?

16) Plane π is given by $2x-y+3z=6$ and line l is given by $x=1-t$, $y=3t$, $z=1+t$. Do l and π intersect? If so, where?

17) Plane π is given by $x - y + z = 3$ and line *l* is given by $\vec{r}(t) = \langle 5 + 2t, 1 - 5t, 3t \rangle$.
Do l and π intersect? If so, where?

18) Two planes π_1 and π_2 are given by $-x + 7y - 2z = 24$ and $-5x + 6y + 3z = -3$, respectively.
l is the line intersection of the planes: $l = \pi_1 \cap \pi_2$. Find the vector equation of line l.

19) Plane π is given by $5x - 3y - 6z = 4$ and line l is given by $\vec{r}(t) = \langle 5 - 10t, 1 + 6t, 12t \rangle$.
Are l and π parallel, perpendicular or neither?

Quadratic Surfaces

Questions:

20) Sketch the graph of the quadric surface $\frac{y^2}{9} + z^2 = 1$.

21) Sketch the graph of the quadric surface $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{6} = 1$.

22) Sketch the graph of the quadric surface $z = \frac{x^2}{4} + \frac{y^2}{4} - 6$.

23) Sketch the graph of the quadric surface $y^2 = 4x^2 + 16z^2$.

24) Sketch the graph of the quadric surface $x = 4 - 5y^2 - 9z^2$.

Answer Key:

1) Onto $xy:(4,7,0)$ Onto $xz:(4,0,-5)$ Onto $yz:(0,7,-5)$

2) a. 5 b. Point P is closer

3) a. $\sqrt{65}$ b. Point Q is closer

4) a.(i) Line (ii) plane b.(i) Circle (ii) Cylinder

5) $\langle -10+11t, 4-8t, 2t\rangle$ $x=-10+11t$, $y=4-8t$, $z=-2t$ $\frac{x+10}{11}-\frac{y-4}{8}=\frac{z}{2}$

6) $\langle -10+4t, 4+3t, -5t\rangle$ $t=\frac{x+10}{4}$, $t=\frac{y-4}{3}$, $t=-\frac{z}{5}$ $\frac{x+10}{4}=\frac{y-4}{3}=-\frac{z}{5}$

7) Perpendicular

8) $v=\frac{-39}{73}$; $u=\frac{88}{73}$

9) $\left(3,-\frac{8}{9},1\frac{5}{9}\right)$

10) a. $(-31,3,0)$ b. Not intersection

11) $2x+2y-3z=-1$

12) $x+3y+4z=2$

13) $4x+8y-2z=-22$

14) Orthogonal

15) Neither

16) $\left(1\frac{1}{2},-1\frac{1}{2},\frac{1}{2}\right)$

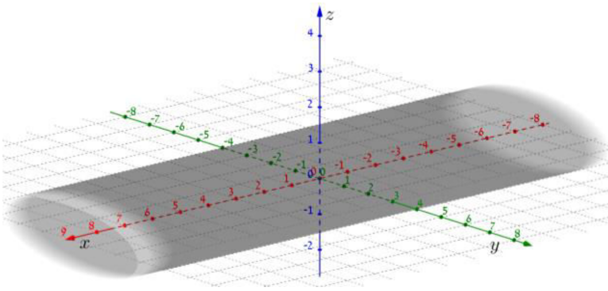
17) Not intersect

18) $\vec{r}(t)=\langle 33t, 2+13t, -5+29t\rangle$

19) Perpendicular

Graphs of Surfaces:

20)



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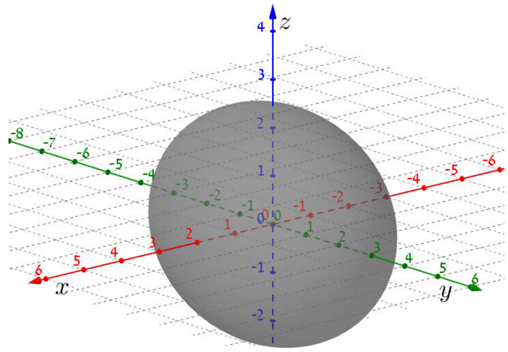

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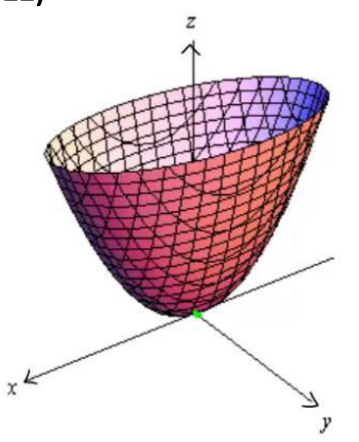

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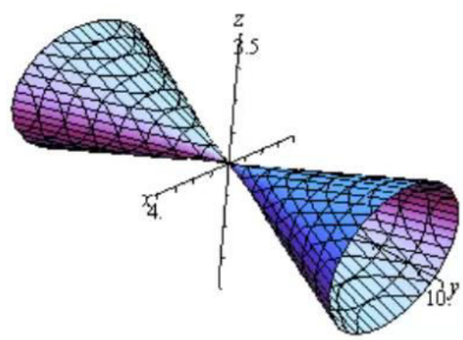

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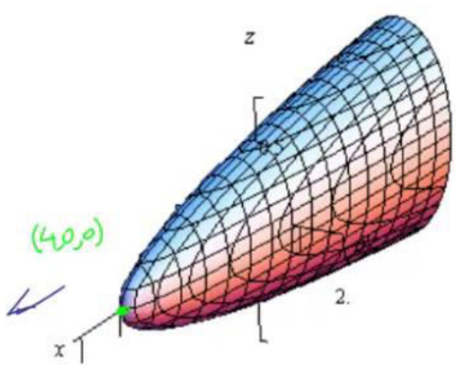


Functions of Several Variables

Questions:

1) Find the domain of the function: $f(x,y)=\sqrt{x^2-2y}$, and try to give a rough sketch.

2) Find the domain of the function: $f(x,y)=\ln(2x-3y+1)$, and try to give a rough sketch.

3) Find the domain of the function: $f(x,y)=\frac{1}{x}+\sqrt{y+4}-\sqrt{x+1}$, and try to give a rough sketch.

4) Find the domain of the function: $f(x,y,z)=\frac{1}{x^2+y^2+4z}$.

5) Sketch a few contours (level curves) for the function: $f(x,y)=x^2+y^2$.

6) Sketch a few contours (level curves) for the implicit function: $2x-3y+z^2=1$.

7) Sketch a few contours (level curves) for the implicit function: $4z+2y^2-x=0$.

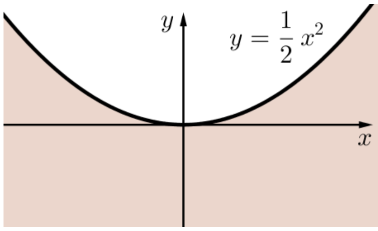
8) Sketch a few contours (level curves) for the implicit function: $y^2=2x^2+z$.

9) Given the implicit function $2x-3y+z^2=1$.

a. Sketch a few traces of the form $x=a$.
b. Sketch a few traces of the form $y=b$.

Answer Key:

1)



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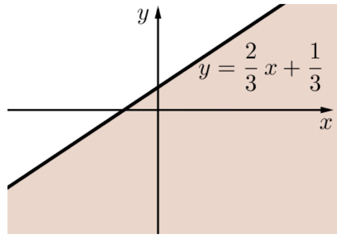

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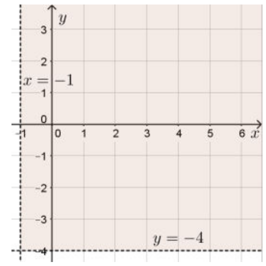
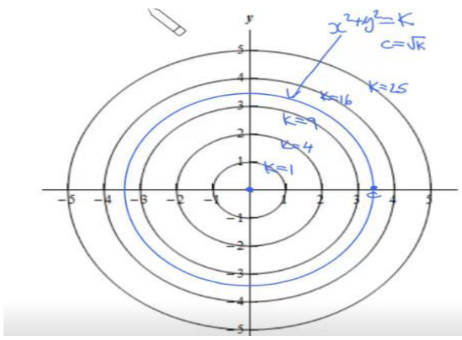


4) z is not on the circular paraboloid: $\frac{z}{-1} = \frac{x^2}{4} + \frac{y^2}{4}$.

5)



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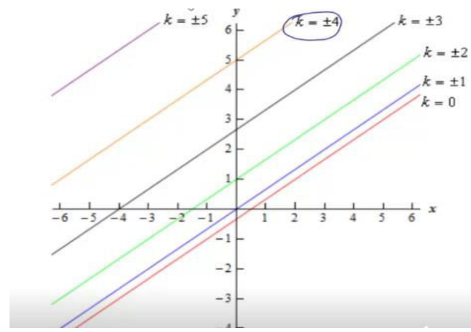

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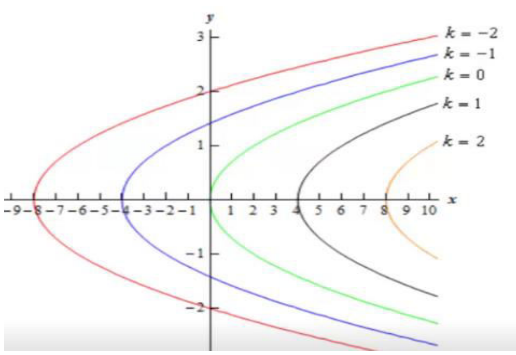

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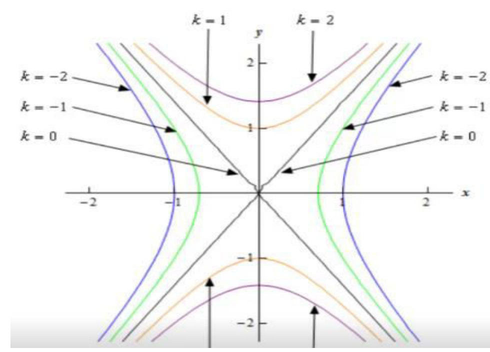

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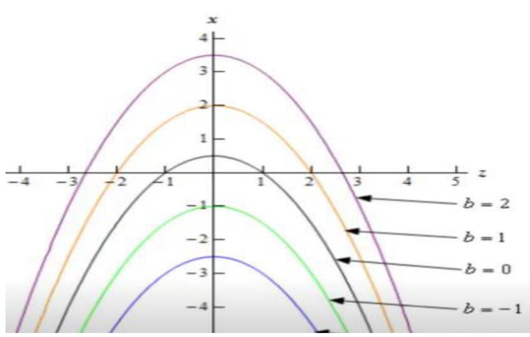


Vector Functions in 3D Coordinates System

Questions:

1) Find the domain of the following vector functions.

a. $\vec{r}(t)=\left\langle t^3-3t,\sqrt{t-1},\frac{5}{t-3}\right\rangle$ (3D)

b. $\vec{r}(t)=\left\langle \sqrt{t+2},\ln\left(9-t^2\right)\right\rangle$ (2D)

2) Sketch the graph of the following 2D vector functions using the specified method:

a. $\vec{r}(t)=\langle 4t,10-2\mathrm{t}\rangle$ by making a table of points.

b. $\vec{r}(t)=\left\langle t+1,\frac{1}{4}t^2+3\right\rangle$ by obtaining an equation in just x and y.

3) Identify the graphs of the following 3D vector functions (do not sketch):

a. $\vec{r}(t)=\langle 2\cos(3t),\sin(3t),4\rangle$

b. $\vec{r}(t)=\langle 5+2t,3-6t,-4-t\rangle$

4) Find the vector equation (function) of the line segment between the two points:

a. $\mathrm{A}(2,4)$, $\mathrm{B}(-3,5)$ (2D)

b. $\mathrm{P}(3,2,0)$, $\mathrm{Q}(8,-5,1)$ (3D)

5) Parametrize, using Polar coordinates, the straight line through the Cartesian points $(3,2)$ and $(1,4)$. The answer should look something like: $\begin{cases} r=r(t) \\ \theta=\theta(t) \end{cases}$, $0\le t\le 1$

Vector Calculus in 3D Coordinates System

Questions:

6) Evaluate the following limits:

a. $\lim_{t\to 2}\left\langle \cos(\pi t), e^{t-2}, \frac{t-2}{t^2-4} \right\rangle$

b. $\lim_{t\to 0}\left\langle (t^3+3)\vec{i} - 2\vec{j} + \frac{1-e^t}{t^2-t}\vec{k} \right\rangle$

c. $\lim_{t\to \infty}\left\langle \frac{3t^2}{t^2-t+3}, e^{-t}, \frac{2}{t^2} \right\rangle$

7) Differentiate the following vector functions:

a. $\vec{r}(t) = (t^3+3)\vec{i} - \sin(2t)\vec{j} + e^{-3t}\vec{k}$

b. $\vec{r}(t) = \langle \ln(\cos t), te^{3t}, 5 \rangle$

c. $\vec{r}(t) = \left\langle \frac{\ln t}{t}, \tan(2t), \sin^2 t \right\rangle$

Tangent, Normal and Binormal Vectors

Questions:

8) Given the vector function $\vec{r}(t) = \langle t^2, \sin 2t, \cos 2t \rangle$.

a. Find the unit tangent vector $\vec{T}(t)$.

b. Find the tangent line, call it $\vec{l}(t)$, at $t = \frac{\pi}{2}$.

9) Given the vector function $\vec{r}(t) = \frac{1}{2}e^{2t}\vec{i} - 2e^t\vec{j} + 2t\vec{k}$.

a. Find the unit tangent vector $\vec{T}(t)$.

b. Find the tangent line, call it $\vec{l}(t)$, at $t = 0$.

10) Given $\vec{r}(t) = \langle 1, \sin 3t, \cos 3t \rangle$, Find the unit normal $\vec{T}(t)$ and the unit binormal $\vec{B}(t)$.

11) Answer the following questions:

a. Compute the frame $\vec{T}(t), \vec{N}(t), \vec{B}(t)$ for the space curve $\vec{r}(t) = (t, 2\sin t, 2\cos t)$

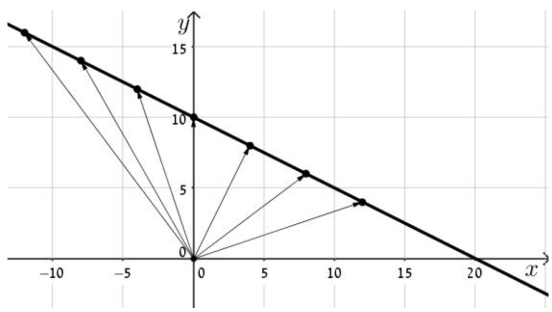
b. Determine the curvature $\kappa(t)$ of $\vec{r}(t)$.

Answer Key:

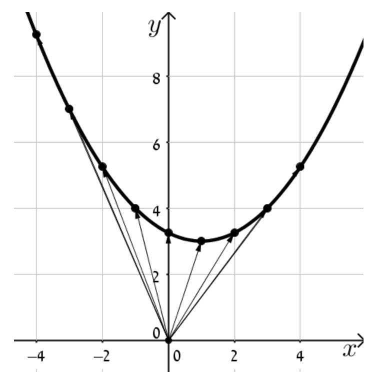
1) a. $t \geq 1$, $t \neq 3$ b. $-2 \leq t < 3$

2) Here are the following figures:

a.



b.



3) a. Ellipse suspended four unites above the xy plane.   b. line

4) a. $\vec{r}(t) = \langle 2-5t, 4+t \rangle \quad 0 \leq t \leq 1$

b. $\vec{r}(t) = \langle 3+5t, 2-7t, t \rangle \quad 0 \leq t \leq 1$

5) $\begin{cases} r(t) = \sqrt{8t^2 - 4t + 13} \\ \theta(t) = \arctan \dfrac{2+2t}{3-2t} \end{cases} \quad 0 \leq t \leq 1$

6) a. $\left\langle 1, 1, \frac{1}{4} \right\rangle$ b. $3\vec{i} - 2\vec{j} + \vec{k}$ c. $\langle 3, 0, 0 \rangle$

7) a. $\vec{r}\,'(t) = 3t^2\vec{i} - 2\cos(2t)\,\vec{j} - 3e^{-3t}\vec{k}$ b. $\vec{r}\,'(t) = \langle -\tan t, e^{3t}(1+3t), 0 \rangle$

c. $r'(t) = \left\langle \dfrac{1-\ln t}{t^2}, \dfrac{2}{\cos^2(2t)}, 2\sin t \cos t \right\rangle$

8) a. $\vec{T}(t) = \left\langle \dfrac{t}{\sqrt{t^2+1}}, \dfrac{\cos 2t}{\sqrt{t^2+1}}, \dfrac{-\sin 2t}{\sqrt{t^2+1}} \right\rangle$ b. $\left\langle \dfrac{\pi^2}{4} + \dfrac{\pi}{2}t, -t, -2 \right\rangle$

9) a. $\vec{T}(t) = \dfrac{e^{2t}}{e^{2t}+2}\vec{i} - \dfrac{2e^2}{e^{2t}+2}\vec{j} + \dfrac{2}{e^{2t}+2}\vec{k}$

b. $\vec{l}(t) = \left(\dfrac{1}{2} + t\right)\vec{i} - (2+2t)\,\vec{j} + 2t\vec{k}$

10) Unit normal: $\vec{N}(t) = \langle 0, -\sin 3t, -\cos 3t \rangle$;   Unit binormal: $\vec{B}(t) = -\vec{i}$

11) a. $\vec{N}(t) = (0, -\sin t, -\cos t)$, $\vec{T}(t) = \dfrac{1}{\sqrt{5}}(1, 2\cos t, -2\sin t)$,

$\vec{B}(t) = \left(\dfrac{2}{\sqrt{5}}, \dfrac{1}{\sqrt{5}}\cos t, \dfrac{-1}{\sqrt{5}}\sin t\right)$ b. $\kappa(t) = \dfrac{2}{5}$

Integrals of Vector Functions

Questions:

1) Evaluate the following integrals:

a. $\int \vec{r}(t)\,dt$ where $\vec{r}(t)=3t^2\vec{i}-\tan(2t)\vec{j}+\frac{3t^2}{t^3-1}\vec{k}$.

b. $\int_1^4 \vec{r}(t)\,dt$ where $\vec{r}(t)=\langle 6te^{3t},4t-3t^2,5\rangle$

Arc Length with Vector Functions

Questions:

2) Find the length of the curve $\vec{r}(t)=\langle 2t-7,3-5t,-6+4t\rangle$, for: $-2\le t\le 5$.

3) Find the length of the curve $\vec{r}(t)=\sqrt{2}t^2\vec{i}+\frac{1}{3}t^3\vec{j}-4t\vec{k}$, for: $0\le t\le 3$.

4) Find the arc length function $s(t)$, for: $\vec{r}(t)=\langle 1+3t^2,4+2t^3\rangle$.

5) Answer the following questions:

a. Find the arc length function $s(t)$, for: $\vec{r}(t)=e^{2t}\cos 2t\vec{i}+2\vec{j}+e^{2t}\sin 2t\vec{k}$.

b. Where on the curve are we, after a travel distance of 10?

Curvatures in 2D and 3D Space

Questions:

6) Find the curvature of the curve: $\vec{r}(t)=\left\langle t,\frac{t^2}{2},t^2\right\rangle$.

7) Find the curvature of the curve: $\vec{r}(t)=3t\,\vec{i}+4\sin t\,\vec{j}+4\cos t\,\vec{k}$.

8) What is the length of the curve: $\vec{r}(t)=\left\langle 2t,\frac{1}{2}t^2,2\ln t\right\rangle$, $1\le t\le e$?

Compute the curvature $\kappa(t)$ of $\vec{r}(t)$.

Velocity and Acceleration in Space

Questions:

9) The acceleration of a body is given by $\vec{a}(t) = 3t\vec{i} - 4e^{-t}\vec{j} + 12t^2\vec{k}$.
The body's initial velocity is $\vec{v}(0) = \vec{j} - 3\vec{k}$ and its initial position is $\vec{r}(0) = -5\vec{i} + 2\vec{j} - 3\vec{k}$.
Find the body's velocity and position functions.

10) The position of a body is given by $\vec{r}(t) = \langle \sin 3t, -\cos 3t, 5 \rangle$.
Determine the tangential and normal components of its acceleration.

11) An alien spaceship is flying through our galaxy, following a trajectory parametrized by $\vec{r}(t) = \langle \sin 2t, 3, \cos 2t \rangle$. Compute its tangential and normal acceleration at $t = \pi$.

12) Suppose that in the superball game this weekend, the Patriots Quarterback Tom Brady wants to throw the football to a receiver standing 40 yards away. If he knows that he can throw the football with an initial speed of 30 yards per second, at what angle with the ground should he throw so that the football arrives in the receiver's hand? What is the flight time of the football? (Disregard air resistance effect, assume Tom and the receiver have the same height, and use $g = 11.25$ yards per square second).

Answer Key:

1) a. $t^3\vec{i}+\frac{1}{2}\ln\left|\cos(2t)\right|\vec{j}+\ln\left|t^3-1\right|\vec{k}+\vec{c}$ b. $\left\langle\frac{1}{3}\left(22e^{12}-4e^3\right),-33,20\right\rangle$

2) $21\sqrt{5}$

3) 21

4) $2\left[\left(1+t^2\right)^{\frac{2}{3}}-1\right]$

5) a. $S(t)=\sqrt{2}\left(e^{2t}-1\right)$ b. $r(t)=e^{2t}\cos 2t\vec{i}+2\vec{j}+e^{2t}\sin 2t\vec{k}$

6) $\kappa=\dfrac{\sqrt{5}}{\left(1+5t^2\right)^{\frac{2}{3}}}$

7) $\dfrac{4}{25}$

8) a. $L=0.5e^2+1.5$ b. $\kappa(t)=\dfrac{2t}{\left(t^2+2\right)^2}$

9) Velocity function: $\vec{v}(t)=\frac{3}{2}t^2\vec{i}+\left(4e^{-t}-3\right)\vec{j}+\left(4t^2-3\right)\vec{k}$

Position function: $\vec{r}(t)=\left(0.5t^3-5\right)\vec{i}+\left(-4e^{-t}-3t+6\right)\vec{j}+\left(t^4-3t-3\right)\vec{k}$

10) $a_T=0$, $a_N=9$

11) $a_T(t)=0$, $\leq a_N(t)=4$

12) $\theta=15°$.

Cylindrical and Spherical Coordinate Systems

Questions:

1) Convert from Cartesian to Cylindrical coordinates:

a. $(3,-7,4)$ b. $(-3,10,4)$

2) Convert the following equation from Cartesian to Cylindrical coordinates: $2y^3-3x^2+4z=7+3y^2$.

3) Convert the following equations from Cylindrical to Cartesian coordinates:

a. $3-r^2=zr+r\cos\theta$ b. $3\cos\theta+5\sin\theta=rz$

4) Identify the surface generated the equation $r^2+6r\sin\theta=13$ in Cylindrical coordinates.
Hint: convert to Cartesian coordinates.

5) Identify the surface generated the equation $z=3-2r^2$ in Cylindrical coordinates.
Hint: convert to Cartesian coordinates.

6) Convert from Cartesian to Spherical coordinates:

a. $(-3,9,4)$ b. $(3,4,-5)$

7) Convert from Spherical to Cylindrical coordinates: $(5,0,\pi)$.

8) Convert from Cylindrical to Spherical coordinates: $(2,1.23,\sqrt{3})$.

9) Convert the following equation from Cartesian to Spherical coordinates: $x^2+y^2-z^2=0$.

10) Convert the following equation from Spherical to Cartesian coordinates: $\rho^2\sin^2\varphi\sin^2\theta+\rho^2\cos^2\varphi=16$.

11) Identify the surface generated by the Spherical equation $\varphi=\frac{3\pi}{4}$.

12) Identify the surface generated by the Spherical equation $\rho=4\sin\varphi\sin\theta$.

Answer Key:

1) a. $\left(\sqrt{58},-1.166,4\right)$ b. $\left(\sqrt{109},1.86°,4\right)$

2) $2r^3\sin^3\theta+4z-3r^2=0$

3) a. $3-\left(x^2+y^2\right)=z\sqrt{x^2+y^2}+x$ b. $3x+5y=\left(x^2+y^2\right)z$

4) Cylinder

5) Circular paraboloid centered on z -axis apex: $(0,0,3)$

6) a. $\left(\sqrt{106},1.892,1.171\right)$ b. $\left(\sqrt{50},0.927,2.356\right)$

7) $(0,0,-5)$

8) $\left(\sqrt{7},1.23,0.428\right)$

9) $\rho^2\left(1-2\cos^2\varphi\right)=0$

10) $y^2+z^2=4^2$

11) Cone facing down with the vertex at the origin and centered along the xz plane

12) Sphere, radius = 2 , center = $(0,2,0)$