

Applications of The Definite Integral – Area and Arc Length

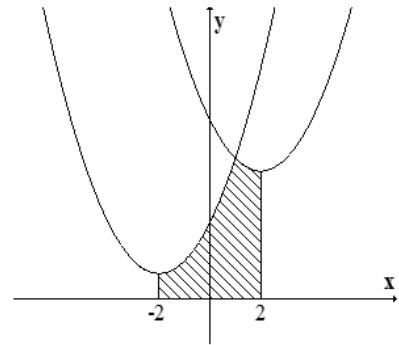
Areas

Questions:

- 1) Given two functions:

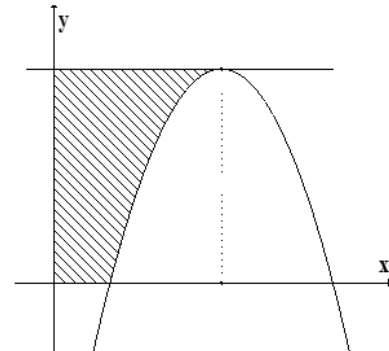
$$f(x) = x^2 + 4x + 6, \quad g(x) = x^2 - 4x + 14.$$

- Find their point of intersection.
- Find the area bounded by the graphs of the two functions, the x -axis, and the lines $x = 2$ and $x = -2$ (the shaded area in the figure).



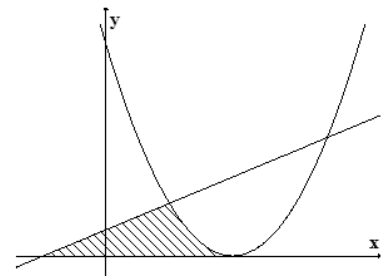
- 2) Given the function $y = -x^2 + 6x - 5$ (see figure).

- Find the coordinates of the maximum point of the function.
- Find the equation of tangent to the graph at its maximum point.
- Find the area bounded by this tangent, by the axes and by the graph of the function (the shaded area in the figure).



- 3) Given the function $f(x) = (x-2)^2$ and the line $y = 0.5x + 0.5$ (see figure).

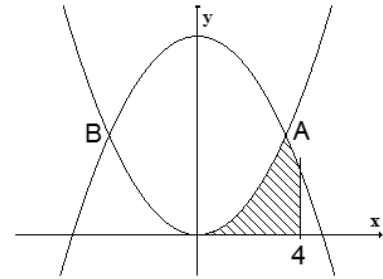
Find the area bounded by the graph of the function, the given line and the x -axis (the shaded area in the figure).



- 4) Given the functions: $f(x) = x^2$, $g(x) = -x^2 + 18$.

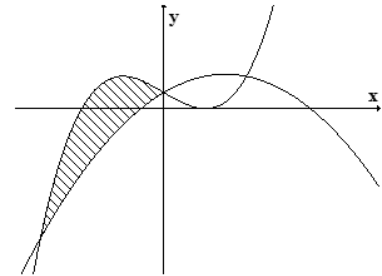
Their graphs intersect at points A and B (see figure).

- Find the x -coordinates of the points A and B.
- Compute the area, in the first quadrant, bounded by the functions, by the x -axis and by the line $x = 4$ (The shaded area in the figure).



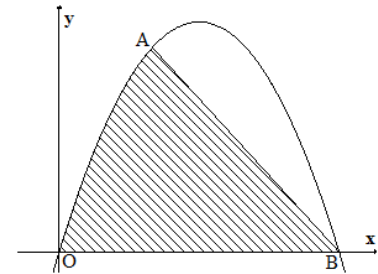
- 5) Given two functions: $y = -x^2 + 3x + 2$, $y = x^3 - 3x + 2$.

- Find the x -coordinates of the intersection points of the graphs of the two functions.
- Compute the area bounded by the graphs of the two functions. (The shaded area in the figure).



- 6) Given the function $f(x) = -x^2 + ax$. The function passes through the point A(2,8) (see figure).

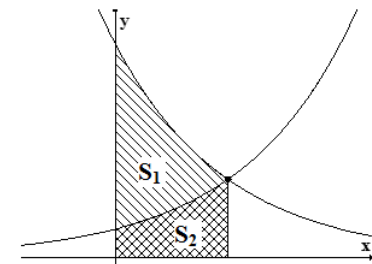
- Find the parameter a .
- The function cuts the x -axis at the origin and at another point B. Find the coordinates of B.
- Compute the area (shaded in the figure) bounded by the graph of the function, by the chord AB and by the x -axis.



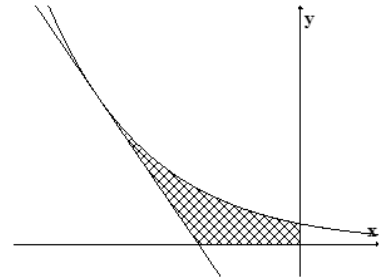
- 7) Given the functions: $f(x) = e^{-x+2}$, $g(x) = e^x$.

- Find the intersection points of the functions with the y -axis.
- Find the intersection point of the two functions.
- The areas S_1 and S_2 are as in the figure.

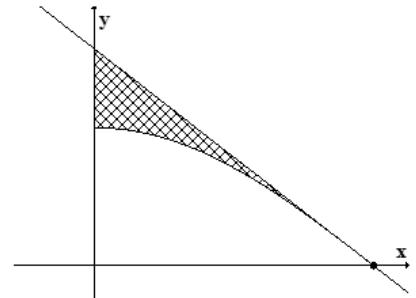
Compute the ratio $\frac{S_1}{S_2}$.



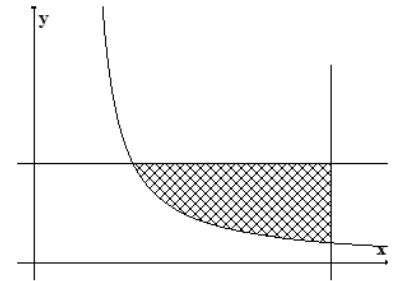
- 8) Given the function $f(x) = e^{-2x}$. A line to the function was drawn at the point $x = -1$ (see figure).
- Find the equation of the tangent.
 - Compute the area (shaded in the figure) bounded by the graph of the function, by the tangent and by the axes.



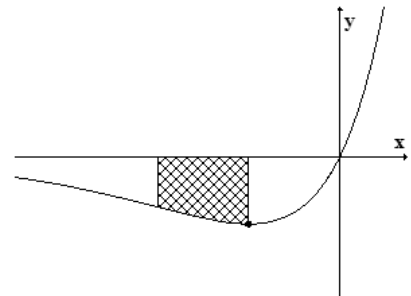
- 9) Given the function $y = \cos 2x$ on the domain $0 \leq x \leq \frac{\pi}{4}$ (see figure). A tangent to the graph is drawn at the point where $x = \frac{\pi}{4}$.
- Find the equation of the tangent.
 - Compute the area (shaded in the figure) bounded by the graph of the function, by the tangent and by the y -axis.



- 10) Compute the area bounded by the graph of the function $y = \frac{1}{2x-1}$ and by the lines $x = 3$ and $y = 1$. (The shaded area in the figure).



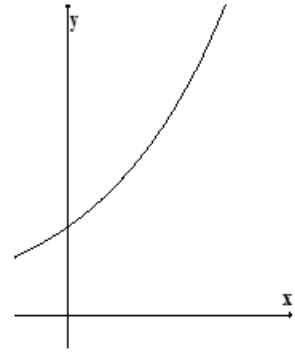
- 11) Given the function $f(x) = e^{2x} - e^x$ which has a minimum as shown in the figure.
- Find the x -coordinate of this minimum point.
 - A perpendicular is dropped from the minimum point to the x -axis. The shaded area in the figure is bounded by this perpendicular, the graph of the function, the x -axis and the line $x = a$. Given that it has an area of $3e^{2a} - e^a$, where $a < \ln 0.5$, find the value of a .



- 12) Given the function $f(x) = e^{\frac{x+1}{2}}$ as in the figure.

The slope of the tangent to the graph at a point A on it is $\frac{e^2}{2}$.

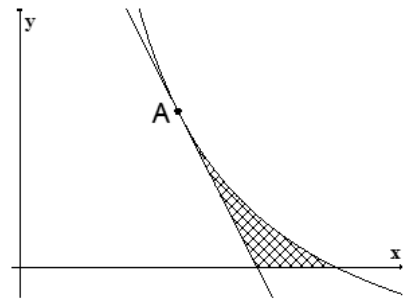
- Find the coordinates of point A.
- Find the equation of the tangent.
- Compute the area bounded by: the graph of the function, the tangent and the y-axis.



- 13) Given the function $f(x) = \frac{8}{x} - 2$ on the domain $x > 0$.

A tangent line to the graph is drawn at the point $A(2, 2)$, (see figure).

- Find the equation of the tangent.
- Compute the area bounded by: the graph of the function, the tangent and the x -axis.



- 14) Answer the following:

- Draw a rough sketch of the graphs of the following 2 functions:

$$f(x) = \sin x ; 0 \leq x \leq \pi$$

$$g(x) = \cos 2x ; 0 \leq x \leq \pi$$

- Shade or highlight the area bounded by these and compute its size.

- 15) Given the function $f(x) = \tan^2 x$ on the domain $-\frac{\pi}{2} < x \leq 0$:

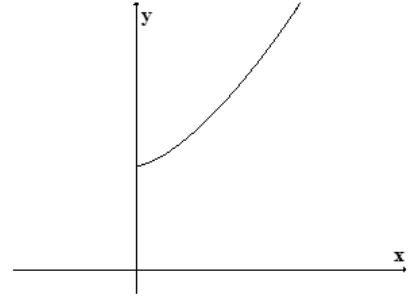
- Find the equation of its tangent at the point where $x = -\frac{\pi}{4}$.
- Show that $\int \tan^2 x dx = \tan x - x + c$ and find the area bounded by: the graph of the function, the tangent and the x -axis.

- 16) Through the point $A(8, 0)$ tangents are drawn to the parabola $y = x^2 - 10x + 25$.

- Find the equations of the tangents.
- Compute the area bounded by the tangents and the parabola.

17) Given the function $f(x) = x\sqrt{x} + 4$ on the domain $x \geq 0$ (see figure).

- Find the equation of the line passing through the origin and tangent to the graph of the given function.
- Compute the area bounded by the graph of the function, the tangent and the y -axis.



18) Answer the following questions:

- Compute the derivative of the function $f(x) = \cos^3 x$.
- Compute the area bounded the x -axis and by the graph of the function $y = \cos^2 x \cdot \sin x$ on the domain $\frac{1}{2}\pi \leq x \leq \frac{3}{2}\pi$.

19) Compute the area bounded by the parabola $y^2 = -x$ and the line $y = x + 6$.

20) Compute the area bounded by the parabola $x = y^2 + 2$ and the line $y = x - 8$.

21) Compute the following integrals:

- $\int_0^a \sqrt{x^2 - a^2} dx$
- $\int_{-a}^a \sqrt{a^2 - y^2} dy$

Curve Length

Questions:

- 1) Find the length of the curve $y = x^{\frac{2}{3}}$ between $x = 1$ and $x = 8$.
- 2) Find the length of the curve $y = \frac{x^4}{8} + \frac{1}{4x^2}$ from $x = 1$ to $x = 2$.
- 3) Find the length of the curve $y = \frac{x^5}{15} + \frac{1}{4x^3}$ from $x = 1$ to $x = 2$.
- 4) Find the length of the curve $y = \frac{2}{3}(1+x^2)^{3/2}$ between $x = 0$ and $x = 3$.
- 5) Find the length of the curve $y = \frac{1}{3}\sqrt{x}(3-x)$ between $x = 0$ and $x = 3$.
- 6) Find the length of the curve $y = \ln x$ from $(1,0)$ to $(2, \ln 2)$.
- 7) Find the length of the curve $y = x^2$ between $x = 1$ and $x = 2$.
 You can use the formula: $\int \sqrt{x^2 \pm a^2} dx = \frac{1}{2}x\sqrt{x^2 \pm a^2} \pm \frac{1}{2}a^2 \ln|x + \sqrt{x^2 \pm a^2}|$.
- 8) Find the length of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$ between $x = 1$ and $x = 8$.
- 9) Find the length of the curve $x = \frac{2}{3}y^{\frac{2}{3}}$ between $y = 0$ and $y = 3$.
- 10) Find the length of the curve $x = \frac{y^4}{4} + \frac{1}{8y^2}$ between $y = 1$ to $y = 2$.
- 11) Find the distance traveled between $t = 0$ and $t = \frac{\pi}{2}$ by a particle $P(x, y)$, Whose position at time t is given by: $x = \cos t + t \sin t$, $y = \sin t - t \cos t$.

12) Find the distance traveled between $t = 0$ and $t = 4$ by a particle $P(x, y)$,

whose position at time t is given by: $x = \frac{t^2}{2}$, $y = \frac{1}{3}(2t+1)^{3/2}$.

13) Find the distance traveled between $t = 0$ and $t = 4$ by a particle $P(x, y)$,

whose position at time t is given by: $x = \frac{1}{3}(2t+3)^{3/2}$, $y = \frac{t^2}{2} + t$.

Final Answers:

1) $l = \frac{(\sqrt{40})^3 - (\sqrt{13})^3}{27} = 7.63$

4) 21

7) 2.32

10) $\frac{123}{32}$

13) 16

2) $2\frac{1}{14}$

5) $2\sqrt{3}$

8) 9

11) $\frac{\pi^2}{8}$

3) $\frac{1097}{480}$

6) 1.22

9) $\frac{14}{3}$

12) 12

Work

Example Questions:

- 1) A spring has a natural length of 20 inches. A 4 *Lbs* force, stretch it to 30 inches. How much work to stretch from 35 to 38 inches?
- 2) A Chain of 2*lbs / ft* is attached to a bucket of coal of 800*lb* at bottom of mine shaft, 500*ft*. Determine the amount of work needed to lift the bucket up the shaft.
- 3) Cylindrical tank is half full with oil of $\rho = 60\text{lb} / \text{ft}^3$. Its radius is 4*ft* and its height is 9*ft*. What is the work done in pumping out the oil to the top of the tank?

Center of mass

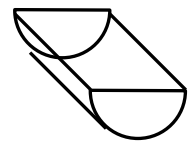
Example questions:

- 1) Find the center of mass of the shape created by the function $f(x) = 4 - x^2$ and the positive rays of the axis. Assume unit weight distribution.
- 2) Find the center of mass of the area between: $y = \sqrt{x}$ and $y = x^2$.

Hydrostatic Pressure and Force:

Example question:

Given a tank of water which is half cylindrical as described.
The radius of $r = 10\text{m}$. Find the hydrostatic force on the bases of the tank.



Mean Value Theorem for Integrals

Question:

Find the point(s) c , as in MTV for Integrals, for $f(x) = \frac{1}{x^2}$ on $[1, 3]$.

Final Answer:

$$c = \sqrt{3}$$

Average Function Value

Question:

Find the average values if the following functions over the given intervals:

- a. $g(x) = \frac{1}{x}$ on $[1, 4]$.
- b. $f(r) = \frac{3}{(1+r)^2}$ on $[1, 6]$.

Final Answers:

- a. $\frac{1}{3} \ln 4$
- b. $\frac{3}{14}$

Numerical Integration

Questions:

- 1) Estimate $\int_0^2 e^{x^2} dx$ using each of the three rules (Midpoint, Trapezoid, Simpson) and $n = 4$.

- 2) Estimate $\int_2^4 \frac{x}{x-1} dx$ using each of the three rules with its specified n .
 - a. Midpoint ($n = 4$)
 - b. Trapezoid ($n = 4$)
 - c. Simpson ($n = 8$)

Final Answers:

- | | | |
|--------------------------|------------------------|---------------|
| 1) Midpoint: 14.48561253 | Trapezoid: 20.64455905 | |
| Simpson: 17.35362645 | | |
| 2) a. 3.08975469 | b. 3.11666667 | c. 3.09872535 |