

Logarithms, Logarithmic Equations, Applications

Rules of Logarithms

Questions:

1) Compute the value of the following logarithms:

a. $\log_5 5$

b. $\log_{125} 5$

c. $\log_{\frac{1}{2}} 16$

2) Compute the value of the following logarithms:

a. $\log_{\frac{1}{4}} \frac{1}{8}$

b. $\log_{\frac{1}{\sqrt[3]{3}}} 9$

c. $\log_{\frac{1}{27}} \sqrt[4]{3}$

3) Compute the value of the following logarithms:

a. $\log_{\frac{1}{\sqrt[3]{25}}} \sqrt[5]{125}$

b. $\log_{0.01} \frac{10}{\sqrt[4]{1000}}$

4) Compute the value of the following expressions:

a. $\log_2 10 + \log_2 6.4$

b. $\log_2 768 - \log_2 6$

c. $\log_{0.25} 80 - \log_{0.25} 5$

5) Compute the value of the following expressions:

a. $3\log_3 6 + \log_3 3.375$

b. $\log_5 50 + \log_5 20 - \log_5 2 - \log_5 4$

6) Compute the value of the following expressions:

a. $-\frac{1}{4}\log_{\sqrt{7}} 81 + 2\log_{\sqrt{7}} 6 - \log_{\sqrt{7}} 84$

b. $\frac{1}{2}\log_{\sqrt[3]{2}} 6 - \frac{1}{2}\log_{\sqrt[3]{2}} 3 + \frac{3}{2}\log_{\sqrt[3]{2}} 4$

7) Compute the value of the following expressions:

a. $\frac{\log_3 16}{\log_3 8}$

b. $\frac{\log_3 6 + 2}{\log_3 108 - \log_3 2}$

8) Compute the value of the expression: $\frac{2 - 3\log 5 + \log 50}{1 + \log 128 - 5\log 2}$.

Hint: convert whole numbers to logarithmic expressions using $n = \log_a a^n$.

9) Compute the value of the following expressions (the logarithm is base 10):

a. $\frac{\log 27}{\log 9}$

b. $\frac{\log 24 - \log 3}{\log 2}$

c. $\frac{1 + \log 5}{\log 2 + 2\log 5}$

10) Prove that the following equality holds (the logarithm is base 10):

$$\frac{\log 9 + 2\log 5 + \log 4}{\log 10 - \log 2 + \log 6} = 2$$

11) Given: $\log_2 7 = a$. Express the following in terms of a :

a. $\log_2 14$

b. $\log_2 49$

12) Given: $\log 4 = a$. Express the following in terms of a :

a. $\log 16$

b. $\log 2$

c. $\log 8$

13) Given: $\log_3 6 = a$, $\log_3 5 = b$. Express the following in terms of a and b :

a. $\log_3 30$

b. $\log_3 1.2$

c. $\log_3 150$

14) Compute the values of the following expressions, using the formula $a^{\log_a b} = b$:

a. $2^{\log_2 3}$

b. $10^{\log 2}$

c. $3^{3\log_3 4}$

15) Compute the values of the following expressions, using the formula $a^{\log_a b} = b$:

a. $8^{\log_2 3}$

b. $\sqrt{6}^{\log_{36} 4}$

c. $\sqrt[3]{5}^{\log_5 64}$

16) Compute the values of the following expressions, using the formula $a^{\log_a b} = b$:

a. $5^{1+\log_5 2}$

b. $\sqrt[3]{27^{1+\log_3 2}}$

17) Compute the following expressions, using the formula $\log_a b = \frac{\log_m b}{\log_m a}$:

a. $\log_3 6 \cdot \log_6 3$

b. $\log_{0.1} 5 \cdot \log_{25} 100$

18) Compute the following expression, using the formula $\log_a b = \frac{\log_m b}{\log_m a}$:

$$\log_{81} 49 \cdot \log_{32} 3 \cdot \log_7 2$$

19) Prove the following equality, using the formula $\log_a b = \frac{\log_m b}{\log_m a}$:

$$\log_3 5 \cdot \log_5 8 + \log_3 2 \cdot \log_2 5 = \log_3 40$$

20) Given: $\log_2 5 = a$. Express the following in terms of a :

a. $\log_5 2$

b. $\log_4 5$

c. $\log_{16} 5$

21) Given: $\log 2 = a$. Express the following in terms of a :

a. $\log 80$

b. $\log_8 40$

c. $\log_{80} 2000$

Answer Key:

1) a. $x=1$ b. $x=\frac{1}{3}$ c. $x=-4$

2) a. $x=\frac{3}{2}$ b. $x=6$ c. $x=-\frac{1}{2}$

3) a. $x=-\frac{9}{10}$ b. $x=-\frac{1}{8}$

4) a. $x=6$ b. $x=7$ c. $x=-2$

5) a. $x=6$ b. $x=3$

6) a. $x=-2$ b. $8\sqrt{2}$

7) a. $\frac{4}{3}$ b. 1

8) 1

9) a. 1.5 b. 3 c. 1

10) Solution in the recording.

11) a. $a+1$ b. $2a$

12) a. $2a$ b. $\frac{1}{2}a$ c. $\frac{3}{2}a$

13) a. $a+b$ b. $a-b$ c. $a+2b$

14) a. 3 b. 2 c. 64

15) a. 27 b. $\sqrt{2}$ c. 4

16) a. 10 b. $\sqrt{216}$

17) a. 1 b. -1

18) $\frac{1}{10}$

19) Solution in the recording.

20) a. $\frac{1}{a}$ b. $\frac{2}{a}$ c. $\frac{a}{4}$

21) a. $3a+1$ b. $\frac{2a+1}{3a}$ c. $\frac{a+3}{3a+1}$

Exponential Equations - Revisited

Questions:

1) Solve the following equations:

a. $2^x = 5$

b. $3^x = 7$

2) Solve the following equations:

a. $2^{x+1} = 5$

b. $3 \cdot 4^{2x+1} = 15$

Answer Key:

1) a. $x = \log_2 5$

b. $x = \log_3 7$

2) a. $x = \log_2 5 - 1$

b. $x = \log_4 5$

Logarithmic Equations - Type 1

Questions:

1) Solve the following equations:

a. $\log_x(2x-4)=1$

b. $\log_x(x^2-3x+15)=2$

2) Solve the following equations:

a. $\log_{\frac{1}{4}}(x^2-4x+11)=-\frac{3}{2}$

b. $\log_{\sqrt{27}}(x^2-8x)=\frac{4}{3}$

3) Solve the following equations:

a. $\log_{\frac{1}{2}}\left(\frac{1}{x^2-4x+11}\right)=3$

b. $\log_x\sqrt{2x+1}=-\frac{1}{2}$

4) Solve the following equations:

a. $\log_{\frac{1}{2}}(\log_{x-2}x)=-1$

b. $\log_{\sqrt{2}}(\log_{x-1}(-5+3x))=2$

5) Solve the following equations:

a. $\log_2(\log_2(x+1))^2=4$

b. $\log_3^2(x+1)=4$

6) Solve the following equations:

a. $\log_7(4+2\log_9x)=0$

b. $\log_{\frac{1}{2}}(\log_2(2\log_3x))=-1$

7) Solve the following equations:

a. $\log_{\frac{1}{2}}(\log_2(\log_{\sqrt{5}}(10x^2+14x)))=-1$

b. $\log_2(\log_{\sqrt{2}}(\log_2(\log_{\sqrt{5}}(2x^2+x-1))))=1$

8) Solve the following equations:

a. $\ln x=4$

b. $2\ln(4x+1)=0$

9) Solve the following equations:

a. $\ln(\ln(x))=1$

b. $\ln(\ln(x+e-1))=0$

10) Solve the following equations:

a. $\log_2(3^x - 17) = 6$

b. $\log_2(12 - 2^{-x}) = 1 - x$

11) Solve the following equations:

a. $3x = -3 + \log_2(12 + 10 \cdot 2^{3x-1})$

b. $2x + 3 = \log_5(30 - 25^{x+1})$

12) Solve the following equations:

a. $x = \frac{1}{2} \log_5(6 \cdot 5^x - 5)$

b. $x = \log_2(68 - 4^{x-1}) + 2$

13) Solve the following equations:

a. $\log_3 x = 2 - \frac{2}{3} \log_3 64$

b. $\log_5(125^x - 24 \cdot 25^x) - 2 = x$

14) Solve the following equations:

a. $4^{\log_4(2x+5)} = 3x - 4$

b. $5^{\log_5(x^2+4x)} = 12$

15) Solve the following equations:

a. $3^{2\log_3(x+1)} = 4$

b. $4^{\frac{1}{\log_4(x+2)}} = \frac{1}{2x+5}$

Answer Key:

- | | |
|--------------------------------|---------------------------|
| 1) a. $x = 4$ | b. $x = 5$ |
| 2) a. $x = 1, 3$ | b. $x = -1, 9$ |
| 3) a. $x = 3, 1$ | b. $x = \frac{1}{2}$ |
| 4) a. $x = 4, 1$ | b. $x = 3$ |
| 5) a. $x = 15, -\frac{15}{16}$ | b. $x = 8, -\frac{8}{9}$ |
| 6) a. $x = \frac{1}{27}$ | b. $x = 9$ |
| 7) a. $x = 1, -2\frac{3}{11}$ | b. $x = 2, -2\frac{1}{2}$ |
| 8) a. $x = e^4$ | b. $x = 0$ |
| 9) a. $x = e^e$ | b. $x = 1$ |
| 10) a. $x = 4$ | b. $x = -2$ |
| 11) a. No solution. | b. $x = -\frac{1}{2}$ |
| 12) a. $x = 1, 0$ | b. $x = 4$ |
| 13) a. $x = \frac{9}{16}$ | b. $x = 2$ |
| 14) a. $x = 9$ | b. $x = 2, -6$ |
| 15) a. $x = 1$ | b. No solution. |

Logarithmic Equations - Type 2

Questions:

1) Solve the following equations:

a. $\log x + \log 2x = \log 36 - \log 2$

b. $\log x^3 + \log \frac{1}{x} = \log 3 + \log 8 - \log 6$

2) Solve the following equations:

a. $2\log_5(x-2) - \log_5 x = \log_5(x-3)$

b. $\log_{\frac{1}{4}} x + 2\log_{\frac{1}{4}}(\sqrt{x} + 3) = \log_{\frac{1}{4}} 16$

3) Solve the following equations:

a. $2\log_2 \sqrt{x} - \log_2 \frac{1}{x+1} = -1 + \log_2 4$

b. $\log(x+2) = \log 3 - \frac{1}{2} \log x^2$

4) Solve the following equations:

a. $\ln 2x^2 + \ln \frac{1}{x} = 2 \ln \sqrt{e}$

b. $\ln 4 + \ln \frac{x}{4} + \ln 2 = \ln x + \ln(x+1)$

5) Solve the following equations:

a. $\log_2 x + 2\log_4 x + 3\log_8 x = \log_2 8$

b. $\log_5 x + \frac{1}{2} \log_{\sqrt{5}} x + 2\log_{25} x = 1.5 \log_{125} x^2$

Answer Key:

1) a. $x = 3$ b. $x = 2$

2) a. $x = 4$ b. $x = 1$

3) a. $x = 1$ b. $x = 1$

4) a. $x = \frac{e}{2}$ b. $x = 1$

5) a. $x = 2$ b. $x = 1$

Logarithmic Equations - Type 3

Questions:

1) Solve the following equations:

a. $2(\log x)^2 + 3\log x - 5 = 0$

b. $4\log^2 x + \log x^3 - 10\log 10 = 0$

2) Solve the following equations:

a. $2\log \sqrt{x} + 2 = \frac{8}{\log x}$

b. $1 + \frac{2}{3\log \sqrt[3]{x}} = \frac{3}{\log^2 x}$

3) Solve the following equations:

a. $\frac{2}{\ln x - 1} + \frac{4}{2 - \ln x^{-1}} = 3$

b. $2\log_2 \frac{1}{\sqrt{x}} + (\log_4 x)^2 = -\frac{3}{4}$

4) Solve the following equations:

a. $\log(x-6) = \log 8 - 3\log \sqrt[3]{x-4}$

b. $\log_x 2 + \log_{2x} 4 = 2\log_{4x} 8$

Answer Key:

1) a. $x = 10, 10^{-2.5}$ b. $x = 17.87, 0.01$

2) a. $x = 100, 0.0001$ b. $x = 10, 0.001$

3) a. $x = e^2, e^{-1}$ b. $x = 8, 2$

4) a. $x = 8$ b. $x = \frac{1}{2}, 2$
 $2^{\frac{1}{3}}$

Logarithmic Equations - Type 4

Questions:

1) Solve the following equations:

a. $x^{\log_4 x} = 4$

b. $x^{\log_2 x} = 16$

2) Solve the following equations:

a. $x^{-1-\log x} = \frac{1}{100}$

b. $(3x)^{2+2\log_3 x} = 9$

3) Solve the following equations:

a. $(x^2)^{\log_4 \frac{1}{x}} = \frac{1}{64x}$

b. $x^{2\log x} = 1000000x^4$

4) Solve the following equations:

a. $0.1^{3-\log(10x)} = \frac{100}{x^3}$

b. $x^{2-\log_5 \frac{x}{5}} = \frac{1}{625}$

5) Solve the following equations:

a. $(10x)^{\frac{2}{\log x}} = 10000x^4$

b. $(\sqrt{x})^{7+\log_3 x} = 9^{1+\log_3 x}$

6) Solve the following equations:

a. $x^{2-\log x^3} = 10\left(\frac{1}{x}\right)^{1+\log x}$

b. $(\sqrt{3x})^{-\log_3(9x)} = x^{-3}$

7) Solve the following equations:

a. $(10x^2)^{\log x-3} = \frac{1}{x^6}$

b. $(4x^2)^{\log_x 4} = x^8$

8) Solve the following equations:

a. $625^{\log_5 x} = 4x^6$

b. $\left(\frac{x^2}{1000}\right)^{-\log x} = 0.01$

Answer Key:

- | | |
|---------------------------------------|-----------------------------------|
| 1) a. $x = 4, \frac{1}{4}$ | b. $x = 16, \frac{1}{16}$ |
| 2) a. $x = 10, \frac{1}{100}$ | b. $x = 1, \frac{1}{9}$ |
| 3) a. $x = 8, \frac{1}{4}$ | b. $x = 1000, \frac{1}{10}$ |
| 4) a. $x = 10$ | b. $x = 625, \frac{1}{5}$ |
| 5) a. $x = \sqrt{10}, \frac{1}{10}$ | b. $x = 3, \frac{1}{81}$ |
| 6) a. $x = 10, \sqrt{10}$ | b. $x = 9, 3$ |
| 7) a. $x = 10, \frac{1}{10\sqrt{10}}$ | b. $x = 2, \frac{1}{\sqrt{2}}$ |
| 8) a. $x = \frac{1}{2}$ | b. $x = 100, \frac{1}{\sqrt{10}}$ |

Applications - Decay and Growth - Version 1

Questions:

- 1) A woman deposited \$500 in a savings account at an interest rate of 3% compounded annually. Determine how much money will be in the account after:
 - a. 5 years.
 - b. 2 years.
 - c. 4 months (Assume all months are exactly $\frac{1}{12}$ of a year).
 - d. 6 months.

- 2) Assume that the population of Earth is growing exponentially at a (constant) rate of 2% per year, and that in 1980 it was 4 billion (4,000,000,000).
 - a. What will the population on Earth be in 2020?
 - b. What was the population on Earth in 1974?
 - c. When a population of 50 billion will be reached?

- 3) The population in a certain city grows exponentially. In a certain year there were 400 thousand residents and 4 years later there were 440 thousand.
 - a. Find the annual growth rate (as a %).
 - b. After how many years (from that certain year) were there 550 thousand residents?

- 4) A man deposited money in the bank at an interest rate of 4% compounded annually. After 5 years he had accumulated \$5000.
 - a. How much did he initially deposit?
 - b. After how many years will he have accumulated \$7000?

- 5) The number of wild animals at a nature reserve grows exponentially. There were 1000 animals at the initial count. At a second count, 20 months later, there were 1400 wild animals.
How many months after the initial count will the reserve have 2000 animals?

- 6) The radioactive isotope carbon-14 decays exponentially with a half-life of 5750 years.
 - a. How many grams of this isotope will remain after 1000 years, if there were 100 grams initially?
 - b. After how many years will there remain just 10 grams of the initial 100 grams?

- 7) In a certain pool there are 240 tons of fish, and the quantity of fish in it increases by 4% each week. In a second pool there are 200 tons of fish, and the quantity of fish in it increases by 10% each week.
 - a. After how many weeks both pools will have the same quantity of fish?
 - b. After how many weeks will the second pool have twice the quantity of fish as the first pool?

- 8) John purchased a car. Assume its value depreciates exponentially. After 4 years its value was \$5,000. After an additional 2 years its value dropped to \$4,000. What was the initial purchase price?
- 9) The value of a share (of stock) is increasing exponentially and doubles itself in 4 years. How long does it take for the share to triple its value?
- 10) Assume the value of a car depreciates exponentially. After 4 years it lost 25% of its value. How many years does it take for the car to lose 50% of its value?

Answer Key:

- 1) a. \$579.64 b. \$530.45 c. \$504.65 d. \$507.44
- 2) a. 8,832,158,654 b. 3,551,885,528 c. 2107 or 2108
- 3) a. 2.41% b. 13.36 years
- 4) a. \$4,109.63 b. 13.5789 years
- 5) 41.2 months
- 6) a. 88.64 grams b. 19,101 years
- 7) a. 3.25 weeks b. 15.61 weeks
- 8) \$7,812
- 9) 6.34 years
- 10) 9.64 years

Applications - Decay and Growth - Version 2

Questions:

- 1) \$10,000 are deposited in an account that earns interest at an annual rate of 4%. Determine how much money will be in the account after 26 months, if the interest is compounded:
 - a. Quarterly.
 - b. Monthly.
 - c. Continuously.

- 2) A bank account is opened with an initial deposit of \$4000. The annual interest rate is 2%. How long will it take for the money to double if the interest is compounded:
 - a. Quarterly.
 - b. Monthly.
 - c. Continuously.

- 3) We deposit \$4500 into an account that earns interest at an annual rate of 8%. How long do we have to wait till there are \$7000 in the account if the interest is compounded:
 - a. Continuously.
 - b. 6 times a year.

- 4) The growth of a colony of bacteria is given by the equation: $Q_t = Q_0 e^{0.215t}$; where t is in hours. Given that there are initially 400 bacteria present.
 - a. How many bacteria will there be after two days?
 - b. How long will it take till there are 20,000 bacteria in the colony?

- 5) A population of bacteria grows exponentially. Initially, there are 400 bacteria present and in 4 days there will be 1,200.
 - a. Determine the growth equation for this population (time is measured in days).
 - b. How long will it take for the population to reach 5,000?

- 6) The number of wild animals at a nature reserve grows exponentially. There were 1000 animals at the initial count. At a second count, 20 months later, there were 1400 wild animals. How many months after the initial count will the reserve have 2000 animals?

- 7) The radioactive isotope carbon-14 decays exponentially with a half-life of 5750 years
 - a. How many grams of this isotope will remain after 1000 years, if there were 100 grams initially?
 - b. After how many years will there remain just 10 grams of the initial 100 grams?

- 8) We initially have 200 grams of a radioactive element and in 950 years 90 grams will remain.
 - a. Determine the exponential decay equation for this element.
 - b. How long will it take for half of the element to decay?
 - c. How long will it take until there is only 1 gram of the element left?

Answer Key:

- 1) a. \$10,900.63 b. \$10,903.76 c. \$10,905.33
- 2) a. 34.74 years. b. 34.68 years. c. 34.65 years.
- 3) a. 5.52 years. b. 5.56 years.
- 4) a. 12,133,303 b. 18.195 hours.
- 5) a. $Q_t = 400 \cdot 3^{\frac{t}{4}}$ b. 9.196 days.
- 6) 41.2 months.
- 7) a. 88.64 grams. b. 19,101 years.
- 8) a. $Q_t = 200e^{\frac{t \ln 0.45}{950}}$ b. 824.65 years. c. 6,303.51 years.