

# Workbook



## Table of Contents

Definite Integrals.....	2
Definite Integrals.....	2
Inequalities.....	3
Riemann Sum .....	4
Fundamental Theorem of Calculus.....	6

# Definite Integrals

## Definite Integrals

### Questions:

Compute the following integrals:

$$1) \int_1^4 (2x^2 - 4x + 1) dx$$

$$2) \int_0^2 \frac{2x+1}{x^2+x+1} dx$$

$$3) \int_2^3 xe^{-x} dx$$

$$4) \int_1^4 \frac{(\ln x)^4}{x} dx$$

$$5) \int_1^{\pi} \cos^2 4x dx$$

$$6) \int_0^4 f(x) ; f(x) = \begin{cases} \sqrt{x} & 0 \leq x \leq 1 \\ \frac{1}{x^2} & x \geq 1 \end{cases}$$

$$7) \int_{-1}^4 \sqrt{4+|x-1|} dx$$

$$8) \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$$

$$9) \int_0^{\pi/2} \frac{\sqrt[4]{\sin x}}{\sqrt[4]{\sin x} + \sqrt[4]{\cos x}} dx$$

10) Let  $f$  be a continuous function. Prove that:

a. If  $f$  is even then  $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$ .

b. If  $f$  is odd then  $\int_{-a}^a f(x) dx = 0$ .

11) Compute the following integrals:

a.  $\int_{-4}^4 \frac{\cos x}{x^3 + x^5} dx$ .

b.  $\int_{-1}^1 \frac{\sin x + 1}{x^2 + 1} dx$ .

**Answer Key:**

- |   |  |                            |
|---|--|----------------------------|
| 1) 13   | 2) $\ln 7$   | 3) $-4e^{-3} + 3e^{-2}$    |
| 4) $\frac{1}{5} \ln^5 4$                            | 5) $\frac{1}{2} \left( \pi - 1 - \frac{1}{8} \sin 8 \right) = 1.062$ | 6) $1 \frac{5}{12}$        |
| 7) $\frac{2}{3} (-16 + 6^{1.5} + 7^{1.5}) = 11.478$ | 8) $\frac{\pi^2}{4}$   | 9) $\frac{\pi}{4} = 0.785$ |
| 10) Refer to the video                              | 11) $\frac{\pi}{2}$  |                            |

---

**Inequalities**

---

**Questions:**

- 1) Prove:  $\frac{2}{41} \leq \int_{-1}^3 \frac{dx}{1+x} \leq 4$ .
- 2) Prove:  $6 \leq \int_{-4}^2 \sqrt{1+x^2} dx \leq 6\sqrt{17}$ .
- 3) Prove:  $2 \leq \int_0^2 e^{x^2} dx \leq 2e^4$ .
- 4) Prove:  $\frac{1}{2} e^{-10} < \int_0^{10} \frac{e^{-x}}{x+10} dx < 1$ .
- 5) Prove:  $\frac{\pi}{14} < \int_0^{\pi/2} \frac{dx}{3+4\sin^2 x} \leq \frac{\pi}{6}$ .
- 6) Compute the integral:  $-\frac{1}{2} \leq \int_0^1 x \sin\left(\frac{\ln x}{x+1}\right) dx \leq \frac{1}{2}$ .
- 7) Prove:  $\int_0^{\pi} x^2 \arctan\left(\frac{\sin x}{x+4}\right) dx \leq \frac{\pi^4}{6}$ .

**Answer Key:**

Refer to the videos.

## Riemann Sum

---

### Questions:

1) Evaluate:  $\lim_{n \rightarrow \infty} \frac{1^4 + 2^4 + 3^4 + \dots + n^4}{n^5}$

2) Evaluate:  $\lim_{n \rightarrow \infty} \left\{ \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+n} \right\}$

3) Evaluate:  $\lim_{n \rightarrow \infty} \left\{ \frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \dots + \frac{n}{n^2+n^2} \right\}$

4) Evaluate:  $\lim_{n \rightarrow \infty} \left\{ \frac{1}{\sqrt{n^2+1^2}} + \frac{1}{\sqrt{n^2+2^2}} + \dots + \frac{1}{\sqrt{n^2+n^2}} \right\}$

5) Evaluate:  $\lim_{n \rightarrow \infty} \left\{ \frac{\sqrt{n+1} + \sqrt{n+2} + \dots + \sqrt{2n}}{n^{3/2}} \right\}$

6) Evaluate:  $\lim_{n \rightarrow \infty} \frac{\sin \frac{1}{n} + \sin \frac{2}{n} + \dots + \sin \frac{n}{n}}{n}$

7) Evaluate:  $\lim_{n \rightarrow \infty} \frac{1 + \sqrt[n]{e} + \sqrt[n]{e^2} + \sqrt[n]{e^3} + \dots + \sqrt[n]{e^{n-1}}}{n}$

8) Evaluate:  $\lim_{n \rightarrow \infty} \sum_{k=1}^n \ln \sqrt[n]{1 + \frac{k}{n}}$

9) Use the Riemann sum definition of integral to evaluate  $\int_0^x x dx$

Hint:  $1+2+3+\dots+n = 0.5n(n+1)$ .

10) Use the Riemann sum definition of integral to evaluate  $\int_0^1 x^2 dx$

Hint:  $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{1}{6}n(n+1)(2n+1)$

11) Use the Riemann sum definition of integral to evaluate  $\int_0^1 x^3 dx$

Hint:  $1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{1}{4}n^2(n+1)^2$

12) Use the Riemann sum definition of integral to evaluate  $\int_0^\pi \sin x dx$

Hint:  $\sin(\alpha) + \sin(2\alpha) + \dots + \sin(n\alpha) = \frac{\sin\left(\frac{n}{2}\alpha\right)\sin\left(\frac{n+1}{2}\alpha\right)}{\sin\left(\frac{\alpha}{2}\right)}$

13) Use the Riemann sum definition of integral to evaluate  $\int_2^5 (2x^2 + 3x) dx$

Hint:  $1 + 2 + 3 + \dots + n = \frac{1}{2}n(n+1)$ ,  $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{1}{6}n(n+1)(2n+1)$

**Answer Key:**

- |                   |                  |                    |                      |                   |
|-------------------|------------------|--------------------|----------------------|-------------------|
| 1) $\frac{1}{5}$  | 2) $\ln 2$       | 3) $\frac{\pi}{4}$ | 4) $\ln(1+\sqrt{2})$ | 5) 1.219          |
| 6) $1 - \cos(1)$  | 7) $1 - \cos(1)$ | 8) $2\ln 2 - 1$    | 9) $\frac{1}{2}$     | 10) $\frac{1}{3}$ |
| 11) $\frac{1}{4}$ | 12) 2            | 13) 109.5          |                      |                   |

## Fundamental Theorem of Calculus

### Questions:

- 1) Recall the Fundamental Theorem of Calculus:  $I(x) = \int_a^x f(t) dt \Rightarrow I'(x) = f(x)$ .

Prove the following generalization:  $I(x) = \int_a^{b(x)} f(t) dt \Rightarrow I'(x) = f(b(x))b'(x)$ .

- 2) We previously generalized the FTC to:  $I(x) = \int_a^{b(x)} f(t) dt \Rightarrow I'(x) = f(b(x))b'(x)$

Generalize again to prove:  $I(x) = \int_{a(x)}^{b(x)} f(t) dt \Rightarrow I'(x) = f(b(x))b'(x) - f(a(x))a'(x)$ .

- 3) Differentiate the following functions:

a.  $I(x) = \int_2^x e^{-t^2} dt$

b.  $I(x) = \int_1^{x^3} \frac{\ln t}{t^2} dt$

c.  $I(x) = \int_2^{x^3+x} t \ln t dt$

d.  $I(x) = \int_{x^3}^{x^2} \frac{dt}{\sqrt{1+t^4}}$

- 4) Evaluate the following limits:

a.  $\lim_{x \rightarrow 0} \frac{\int_0^x \frac{t dt}{\cos t}}{\sin^2 x}$

b.  $\lim_{x \rightarrow 0} \frac{1}{x^3} \int_0^{x^2} \sin \sqrt{t} dt$

c.  $\lim_{x \rightarrow 4} \frac{x}{x-4} \int_4^x e^{t^2} dt$

- 5) Investigate the function  $F(x) = \int_0^x (t+1)^4 (t-1)^{10} dt$ , particularly the following features:

- Domain of definition
- Extrema and intervals of increase/decrease
- Inflection points and intervals of concavity/convexity

**Answer Key:**

1) Refer to the video

2) Refer to the video

3) a.  $I'(x) = e^{-x^2}$       b.  $\frac{9 \ln x}{x^4}$       c.  $I'(x) = (x^3 + x) \ln(x^3 + x)(3x^2 + 1)$

d.  $I'(x) = \frac{2x}{\sqrt{1+x^8}} - \frac{3x^2}{\sqrt{1+x^{12}}}$

4) a.  $\frac{1}{2}$       b.  $\frac{2}{3}$       c.  $4e^{16}$

5) a. All  $x$       b. No extrema, the function is always increasing

c. Inflection points:  $-1, -\frac{3}{7}, 1$

concave:  $x < -1$  or  $-\frac{3}{7} < x < 1$

convex:  $-1 < x < -\frac{3}{7}$  or  $x > 1$