proprep™

Workbook

Table of Contents

Inner Product Spaces

Inner Product Spaces

Questions:

1) For each two vectors $u=[x_1,x_2]$, $v=[y_1,y_2]$ in $\mathbb{R}^2$ we define

$\langle u,v\rangle = x_1y_1 - 3x_1y_2 - 3x_2y_1 + 4x_2y_2$.

Check if this defines an inner product on $\mathbb{R}^2$.

2) For each two vectors $u=[x_1,x_2]$, $v=[y_1,y_2]$ in $\mathbb{R}^2$ we define

$\langle u,v\rangle = x_1y_1 - 3x_1y_2 - 3x_2y_1 + kx_2y_2$.

For which values of the parameter k does the above define an inner product on $\mathbb{R}^2$?

3) For each two vectors $u=[x_1,x_2,x_3]$, $v=[y_1,y_2,y_3]$ in $\mathbb{R}^3$ we define

$<u,v>= x_1y_1 + kx_1y_3 + x_2y_2 + kx_3y_1 + x_3y_3$.

For which values of the parameter k does the above define an inner product on $\mathbb{R}^3$?

4) For each two vectors $u=[x_1,x_2,...,x_n]$, $v=[y_1,y_2,...,y_n]$ in $\mathbb{R}^n$ we define

$<u,v>=\sum_{i=1}^{n} k_i x_i y_i$, where the parameters $k_1,...,k_n$ are positive numbers.

Show that the above definition gives an inner product on $\mathbb{R}^n$.

What do we get if $k_i = 1$ for all $1 \le i \le n$?

5) For each two matrices A, B in $M_{m\times n}[\mathbb{R}]$ we define $\langle A,B\rangle = tr(B^T A)$.

Check if this defines an inner product on $M_{m\times n}[\mathbb{R}]$.

6) For each two functions f, g in $C[a,b]$ we define $\langle f,g\rangle = \int_a^b f(x)\cdot g(x)dx$.

Check if this defines an inner product on $C[a,b]$.

Answer Key:

1) Does not define.

2) $k > 9$

3) $-1 < k < 1$

4) Solution in the recording.

5) Solution in the recording.

6) Solution in the recording.

Norm and Distance

Questions:

1) Take the IPS $\mathbb{R}^3$, with the standard inner product*, and take the three vectors:
$u=[1,-2,2]$, $v=[3,-2,6]$, $w=[5,3,-2]$, in $\mathbb{R}^3$.
Compute the following:

a. $\langle u,v\rangle$ b. $\langle u,w\rangle$ c. $\langle v,w\rangle$ d. $\langle u+v,w\rangle$ e. $\|u\|$

f. $\|v\|$ g. $\|u+v\|$ h. $d(u,v)$ i. $\hat{u}$ j. $\hat{v}$

*AKA dot product and we can write $u\cdot v$ instead of $\langle u,v\rangle$

2) We are given three matrices $A=\begin{bmatrix}10 & 9 & 8\\ 7 & 6 & 5\end{bmatrix}$, $B=\begin{bmatrix}2 & 3 & 4\\ 5 & 6 & 7\end{bmatrix}$, $C=\begin{bmatrix}3 & -5 & 2\\ 1 & 0 & -4\end{bmatrix}$

in the IPS $M_{2\times3}[\mathbb{R}]$, with inner product defined by $\langle X,Y\rangle = tr\left(Y^T X\right)$.

Compute the following:

a. $\langle A,B\rangle$ b. $\langle A,C\rangle$ c. $\langle A,B+C\rangle$ d. $\langle B,C\rangle$

e. $\langle 4A+10B,11C\rangle$ f. $\|A\|$ g. $\|B\|$ h. $d(A,B)$ i. $\hat{A}$

3) We are given three functions $p(x)=x+3$, $q(x)=3x+1$, $r(x)=x^2-4x-1$

in the IPS $C[0,1]$, with inner product $\langle f,g\rangle=\int_0^1 f(x)\cdot g(x)\,dx$.

Compute:

a. $\langle p,q\rangle$ b. $\langle p,r\rangle$ c. $\langle p,q+r\rangle$ d. $\|p\|$ e. $d(p,q)$ f. $\hat{r}$

4) Prove: $\|u+v\|^2=\|u\|^2+2\langle u,v\rangle+\|v\|^2$.

5) Prove: $\|u-v\|^2=\|u\|^2-2\langle u,v\rangle+\|v\|^2$.

6) Prove: $\langle u-v,u+v\rangle=\|u\|^2-\|v\|^2$.

7) Prove: $\|u+v\|^2 + \|u-v\|^2 = +2\|u\|^2 + 2\|v\|^2$.

Give a geometric interpretation in the plane.

8) Prove: $\frac{1}{4}\left(\|u+v\|^2 - \|u-v\|^2\right) = \langle u,v \rangle$.

Answer Key:

1) a. 19 b. -5 c. -3 d. -8

a. 19 b. -5 c. -3 d. -8 e. 3

f. 7 g. $\sqrt{96}$ h. $\sqrt{20}$ i. $\left[\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right]$ j. $\left[\frac{3}{7}, -\frac{2}{7}, \frac{6}{7}\right]$

1)

a. 185 b. -12 c. 173 d. -24 e. -3168

f. $\sqrt{355}$ g. $\sqrt{139}$ h. $\sqrt{124}$ i. $\hat{A} = \frac{1}{\sqrt{355}} \cdot \begin{bmatrix} 10 & 9 & 8 \\ 7 & 6 & 5 \end{bmatrix}$

2)

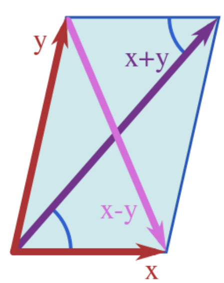
a. 9 b. -9.583333 c. -0.58333

d. $\sqrt{\frac{37}{3}}$ e. $\sqrt{\frac{4}{3}}$ f. $\hat{r} = \frac{r}{\|r\|} = \frac{x^2 - 4x - 1}{\sqrt{7\frac{13}{15}}}$

4-6) Solution in the recording.

7) Solution in the recording.

Geometric interpretation:



8) Solution in the recording.

Orthogonality

Orthogonality

Questions:

1) Prove that the vectors $u=[1,2,3]$, $v=[4,7,-6]$ are orthogonal in $\mathbb{R}^3$.

2) Find the value of the parameter k, for which the vectors $u=[1,k,3]$, $v=[4,7,-6]$, are orthogonal in $\mathbb{R}^3$.

3) Find a unit vector perpendicular to the vectors $u=[1,2,3]$, $v=[2,5,7]$, in $\mathbb{R}^3$.

4) Show that the polynomials $p(x)=2x-1$, $q(x)=6x^2-6x+1$, are orthogonal in $C[0,1]$, with the inner product $\langle f,g\rangle=\int_0^1 f(x)\cdot g(x)dx$.

5) In the space $P_n[\mathbb{R}]$ (polynomials with degree $\le n$, over $\mathbb{R}$), we define an inner product as follows: $\langle p,q\rangle=\sum_{k=0}^{n} p(k)q(k)=p(0)q(0)+p(1)q(1)+...+p(n)q(n)$.
Show that the polynomials $p(x)=x(x-2)(x-4)(x-6)$, $q(x)=(x-1)(x-3)(x-5)(x-7)$ are orthogonal in $P_7[\mathbb{R}]$ with the inner product defined above.

6) Given two matrices $A=\begin{bmatrix} k & 1 \\ 3 & -1 \end{bmatrix}$, $B=\begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix}$, in $M_{2\times2}[\mathbb{R}]$, with inner product $\langle X,Y\rangle=tr\left(Y^T X\right)$.
For which value(s) of k are these matrices orthogonal?

7) Prove that $\|u+v\|=\|u-v\| \Leftrightarrow u\perp v$. Give a geometric interpretation in $\mathbb{R}^2$.

8) Prove that $\|u+v\|^2=\|u\|^2+\|v\|^2 \Leftrightarrow u\perp v$. Give a geometric interpretation in $\mathbb{R}^2$.

9) Prove that $\|u\|=\|v\| \Rightarrow (u-v)\perp(u+v)$. Give a geometric interpretation in $\mathbb{R}^2$.

Answer Key:

1) Solution in the recording.

2) $k = 2$

3) $\hat{w} = \frac{w}{\|w\|} = \frac{[-1,-1,1]}{\sqrt{1+1+1}} = \left[\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]$

4) Solution in the recording.

5) Solution in the recording.

6) There are no such values.

7) Solution in the recording.

8) Solution in the recording.

9) Solution in the recording.

Orthogonal Complement

Questions:

1) Let $W = span\{[1,2,-1,1],[2,5,3,1]\}$, in $\mathbb{R}^4$ (standard inner product).

a. Find a basis and the dimension of $W^\perp$.
b. Show that the orthogonal decomposition theorem is satisfied.

2) Let $W = span\{[1,1,1]\}$ in $\mathbb{R}^3$.

a. Find a basis and the dimension of $W^\perp$.
b. Show that the orthogonal decomposition theorem is satisfied.

3) Consider the IPS $P_2[\mathbb{R}]$ with the inner product "borrowed" from $C[0,1]$:

$\langle p,q\rangle = \int_0^1 p(x)\cdot q(x)dx$.

Let $W = span\{x\} \subseteq P_2[\mathbb{R}]$. Find a basis and the dimension of $W^\perp$.

4) Consider the IPS $P_2[\mathbb{R}]$ with the inner product "borrowed" from $C[0,1]$:

$\langle p,q\rangle = \int_0^1 p(x)\cdot q(x)dx$

Let $W = span\{x,x^2\} \subseteq P_2[\mathbb{R}]$. Find a basis and the dimension of $W^\perp$.

5) Let $W = span\left\{\begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix},\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\right\}$ in $M_{2\times 2}[\mathbb{R}]$, with inner product $\langle A,B\rangle = tr\left(B^T A\right)$.

Find a basis and the dimension of $W^\perp$.

6) Find the orthogonal complement, $W^\perp$, to the subspace W of 3×3 diagonal matrices in $M_3[\mathbb{R}]$ (with its usual inner product).

7) Find a basis for the orthogonal complement of the space, W, of symmetric 2×2 matrices, as a subspace of $M_2[\mathbb{R}]$ (with the usual inner product).

8) Suppose we're given a homogeneous $m\times n$ SLE: $A\cdot \underline{x} = \underline{0}$ (in matrix notation). Let U be the solution space of the system in $\mathbb{R}^n$ (with usual inner=dot product). Describe U using the concept of orthogonal complement and the concept of the row space of the matrix A.

9) Let W_1, W_2 be subsets os an IPS V.
Prove that $W_1 \subseteq W_2 \Rightarrow W_2^{\perp} \subseteq W_1^{\perp}$.

10) Let *W* be a subspace of *V* (an inner product space).
Prove that $W \subseteq W^{\perp\perp}$.

11) Let W be a subspace of V (an inner product space).
Assume that V has finite dimension.
Prove that $W = W^{\perp\perp}$.

12) Suppose that W_1, W_2 are subspaces of V (an IPS).
Prove that $(W_1 + W_2)^{\perp} = W_1^{\perp} \cap W_2^{\perp}$.

13) Suppose that W_1, W_2 are subspaces of *V* (an IPS).
Prove that $(W_1 \cap W_2)^{\perp} = W_1^{\perp} + W_2^{\perp}$.

Answer Key:

1) a. $W^{\perp} = span\{[-3,1,0,1],[11,-5,1,0]\}$ and $\dim(W^{\perp}) = 2$.

b. Solution in the recording.

2) a. $W^{\perp} = span\{[-1,0,1],[-1,1,0]\}$ and $\dim(W^{\perp}) = 2$.

b. Solution in the recording.

3) $W^{\perp} = span\left\{-\frac{2}{3}+x, -\frac{1}{2}+x^2\right\}$ and $\dim(W^{\perp}) = 2$.

4) $W^{\perp} = span\{3x^2 - 12x + 10\}$ and $\dim(W^{\perp}) = 1$.

5) $W^{\perp} = span\left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\right\}$ and $\dim(W^{\perp}) = 2$..

6) $$B_{W^{\perp}} = \left\{\begin{matrix}\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix},\\ \begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\1&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix}\end{matrix}\right\}.$$

7) $B_{W^{\perp}} = \left\{\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\right\}$.

8) Solution in the recording.

9) Solution in the recording.

10) Solution in the recording.

11) Solution in the recording.

12) Solution in the recording.

13) Solution in the recording.

Orthogonal Sets and Bases

Questions:

1) Given the set of vectors $S=\{[2,1,-4],[1,2,1],[3,-2,1]\}$ in $\mathbb{R}^3$.

g. Show that the set S is orthogonal.

h. Normalize vectors in S to obtain an orthonormal set.

i. Without computation, prove that S is a basis of $\mathbb{R}^3$.

2) Given the set of vectors $S=\{[2,1,-4],[1,2,1],[3,-2,1]\}$ in $\mathbb{R}^3$.

Write the vector $[13,-1,7]$ as a linear combination of the members of S

by using inner products (and NOT row operations, echelon form, etc.).

3) Given the set of vectors $S=\{[2,1,-4],[1,2,1],[3,-2,1]\}$ in $\mathbb{R}^3$.

Write the coordinate vector of a general $[a,b,c]$ in $\mathbb{R}^3$ relative to S,

by using inner products and orthogonality.

Hint: We showed in an earlier exercise that S is an orthogonal basis of $\mathbb{R}^3$.

4) Suppose that $B=\{u_1,u_2,...,u_n\}$ is an orthogonal basis of V.

Prove that for all $v\in V$ $v=\frac{<v,u_1>}{<u_1,u_1>}u_1+\frac{<v,u_2>}{<u_2,u_2>}u_2+...+\frac{<v,u_n>}{<u_n,u_n>}u_n$.

Remark: the constants $a_i=\frac{<v,u_i>}{<u_i,u_i>}$ $(i=1,2,\ldots,n)$ are called the *Fourier coefficients*

[or *components*] of v relative to B.

5) Let $V=C[0,\pi]$ with the integral inner product and consider the set

$S=\{\cos x,\cos 2x,\cos 3x,...\}$ in V.

Is S orthogonal? If so, is it orthonormal?

If it's orthogonal but not orthonormal, then normalize it.

6) Let $V=C[0,2\pi]$ with the integral inner product and consider the set

$S=\{1,\cos x,\sin x,\cos 2x,\sin 2x,...\}$ in V. Is S orthogonal? If so, is it orthonormal?

If it's orthogonal but not orthonormal, then normalize it.

7) Given the set $S=\{[2,4,4],\ [4,-1,-1],\ [0,2,-2]\}$ in $\mathbb{R}^3$ (usual inner product).

Is S an orthogonal set? If so, is it: a. orthonormal? b. a basis of $\mathbb{R}^3$?
If it's orthogonal but not orthonormal, then normalize it.

8) Given the set $S=\{1,x,x^2,x^3\}$ in $P_3[\mathbb{R}]$ with the integral inner product on $[0,1]$.

Is S an orthogonal set? If so, is it: a. orthonormal? b. a basis of $P_3[\mathbb{R}]$?

If it's orthogonal but not orthonormal, then normalize it.

9) Given $S=\{1,2x-1,6x^2-6x+1\}$ in $P_2[\mathbb{R}]$ with the integral inner product on $[0,1]$.

Is S an orthogonal set? If so, is it: a. orthonormal? b. a basis of $P_2[\mathbb{R}]$?

If it's orthogonal but not orthonormal, then normalize it.

10) Given a set $S=\left\{\begin{bmatrix}2&4&6\\0&2&4\\0&0&2\end{bmatrix},\begin{bmatrix}1&0&0\\0&-1&0\\0&0&0\end{bmatrix},\begin{bmatrix}1&1&-1\\0&1&-1\\0&0&1\end{bmatrix}\right\}\subseteq M_3[\mathbb{R}]$

(standard inner product).

Is S an orthogonal set? If so, is it:

a. orthonormal?

b. a basis of $M_3[\mathbb{R}]$?

If it's orthogonal but not orthonormal, then normalize it.

Answer Key:

1) a-c. Solution in the recording.

2) $[13,-1,7]=-\frac{1}{7}[2,1,-4]+3[1,2,1]+\frac{24}{7}[3,-2,1]$

3) $[a,b,c]=\frac{2a+b-4c}{21}[2,1,-4]+\frac{a+2b+c}{6}[1,2,1]+\frac{3a-2b+c}{14}[3,-2,1]$

4) Solution in the recording.

5) It's orthonormal but not orthogonal; $\hat{S}=\left\{\frac{\cos x}{\sqrt{0.5\pi}},\frac{\cos 2x}{\sqrt{0.5\pi}},\frac{\cos 3x}{\sqrt{0.5\pi}},...\right\}$.

6) It's orthogonal but not orthonormal; $\hat{S}=\left\{\frac{1}{\sqrt{2\pi}},\frac{\cos x}{\sqrt{\pi}},\frac{\sin x}{\sqrt{\pi}},\frac{\cos 2x}{\sqrt{\pi}},\frac{\sin 2x}{\sqrt{\pi}},...\right\}$.

7) It's an orthogonal set, but (a.) not orthonormal, and (b.) it's a basis.

$$\hat{S}=\left\{\frac{1}{\sqrt{36}}[2,4,4],\frac{1}{\sqrt{8}}[0,2,-2],\frac{1}{\sqrt{18}}[4,-1,-1]\right\}$$

8) It's not orthogonal.

9) It's orthogonal, but (a.) not orthonormal, and (b.) it's a basis.

$$\hat{S}=\left\{1,\sqrt{3}(2x-1),\sqrt{5}\left(6x^2-6x+1\right)\right\}$$

10) It's an orthogonal set, but (a.) not orthonormal and also (b.) not a basis.

$$\hat{S}=\left\{\frac{1}{\sqrt{80}}\begin{pmatrix}2&4&6\\0&2&4\\0&0&2\end{pmatrix},\frac{1}{\sqrt{2}}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix},\frac{1}{\sqrt{6}}\begin{pmatrix}1&1&-1\\0&1&-1\\0&0&1\end{pmatrix}\right\}$$