

Workbook



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Linear Transformations

Linear Transformation Definition

Questions:

Check if the following transformations are linear transformations:

- 1) $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$; $T[x, y] = [x + y, x - y]$
- 2) $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$; $T[x, y, z] = [x + y - 2z, x + 2y + z, 2x + 2y - 3z]$
- 3) $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$; $T[x, y, z] = [2x + z, |y|]$
- 4) $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$; $T[x, y] = [xy, y, z]$
- 5) $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$; $T[x, y, z] = [x + 1, x + y, x + z]$
- 6) $T: M_n[\mathbb{R}] \rightarrow M_n[\mathbb{R}]$; $T(A) = BA + AB$, where $B \in M_n[\mathbb{R}]$ is some fixed matrix.
- 7) $T: M_n[\mathbb{R}] \rightarrow M_n[\mathbb{R}]$; $T(A) = A + A^T$
- 8) $T: M_n[\mathbb{R}] \rightarrow M_n[\mathbb{R}]$; $T(A) = |A| \cdot I$ ($|A|$ is the determinant of A)
- 9) $T: M_n[\mathbb{R}] \rightarrow M_n[\mathbb{R}]$; $T(A) = A \cdot A^T$
- 10) $T: M_n[\mathbb{R}] \rightarrow M_n[\mathbb{R}]$; $T(A) = A^3$
- 11) $T: P_3[\mathbb{R}] \rightarrow P_2[\mathbb{R}]$; $T(a + bx + cx^2 + dx^3) = a + bx + cx^2$
- 12) $T: P_n[\mathbb{R}] \rightarrow P_n[\mathbb{R}]$; $T(p(x)) = p(x + 1)$
- 13) $T: P_n[\mathbb{R}] \rightarrow P_n[\mathbb{R}]$; $T(p(x)) = p'(x) + p''(x)$
- 14) $T: P_n[\mathbb{R}] \rightarrow P_{2n}[\mathbb{R}]$; $T(p(x)) = p^2(x)$

15) Check if the following transformations are linear transformations:

a. $T: \mathbb{C}[\mathbb{R}] \rightarrow \mathbb{C}[\mathbb{R}] ; T(z) = \bar{z}$

b. $T: \mathbb{C}[\mathbb{C}] \rightarrow \mathbb{C}[\mathbb{C}] ; T(z) = \bar{z}$

16) For which value(s) of m is the following a linear transformation?

$$T: \mathbb{R}^2 \rightarrow \mathbb{R}^2 ; T[x, y] = [m^2 x^{2m}, y^{2m} + x]$$

17) Is there a linear transformation $T: \mathbb{R}^4 \rightarrow \mathbb{R}^3$, such that:

$$T[1, 2, -1, 0] = [0, 1, -1], \quad T[-1, 0, 1, 1] = [1, 0, 0], \quad T[0, 4, 0, 2] = [2, 2, -2]?$$

If there isn't, explain why.

If there is, find such a T and say whether this T is unique.

18) Is there a linear transformation $T: P_2[\mathbb{R}] \rightarrow P_2[\mathbb{R}]$, such that:

$$T(1) = 4, \quad T(4x + x^2) = x, \quad T(1 - x) = x^2 + 1?$$

If there isn't, explain why.

If there is, is it unique? If T is unique, find its formula.

19) Is there a linear transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$, such that:

$$T[1, 1, 0] = [1, 2, 3], \quad T[0, 1, 1] = [4, 5, 6], \quad T[0, 0, 1] = [7, 8, 9]?$$

If there isn't, explain why.

If there is, is it unique? If T is unique, find its formula.

20) Is there a linear transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$, such that:

$$T[1, 0, 1] = [1, 1, 0], \quad T[0, 1, 1] = [1, 2, 1], \quad T[0, 0, 1] = [0, 1, 1]?$$

If there isn't, explain why.

If there is, is it unique? If T is unique, find its formula.

Answer Key:

- 1) T is linear.
- 2) T is linear.
- 3) T is not linear.
- 4) T is not linear.
- 5) T is not linear.
- 6) T is linear.
- 7) T is linear.
- 8) T is not linear.
- 9) T is not linear.
- 10) T is not linear.
- 11) T is linear.
- 12) T is linear.
- 13) T is linear.
- 14) T is not linear.
- 15) a. T is linear.
b. T is not linear.
- 16) $m = 2$
- 17) Yes, not unique $T[x, y, z, t] = [\frac{1}{2}y - x, \frac{1}{2}y, -\frac{1}{2}y]$.
- 18) Yes, unique $T \bar{T}[a, b, c] = [4a + 3b - 12c, c, 4c - b]$.
- 19) Yes, unique $T T[x, y, z] = [4x - 3y + 7z, 5x - 3y + 8z, 6x - 3y + 9z]$.
- 20) Yes, unique $T T[x, y, z] = [x + y, y + z, z - x]$.

Image and kernel

Questions:

1) Let $T: \mathbb{R}^4 \rightarrow \mathbb{R}^3$ be the linear transformation defined by $T[x, y, z, t] = [x + y, y - 4z + t, 4x + y + 4z - t]$.

- Find a basis and the dimension of the kernel of T .
- Find a basis and the dimension of the image of T .

2) Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^4$ be the linear transformation defined by $T[x, y, z] = [x - 4y - z, x + y, y - z, x + 4z]$.

- Find a basis and the dimension of the kernel of T .
- Find a basis and the dimension of the image of T .

3) Let $T: \mathbb{R}^4 \rightarrow \mathbb{R}^3$ be the linear transformation defined by

$$T[x, y, z, t] = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & -2 \\ 2 & 6 & 10 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}.$$

- Find a basis and the dimension of the kernel of T .
- Find a basis and the dimension of the image of T .

4) Let $T: M_2[\mathbb{R}] \rightarrow M_2[\mathbb{R}]$ be the linear transformation defined by

$$T(A) = A \cdot \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \cdot A.$$

- Find a basis and the dimension of the kernel of T .
- Find a basis and the dimension of the image of T .

5) Let $T: P_2[\mathbb{R}] \rightarrow P_2[\mathbb{R}]$ be the linear transformation defined by

$$T(p(x)) = p(x+1) - p(x+4).$$

- Find a basis and the dimension of the kernel of T .
- Find a basis and the dimension of the image of T .

6) Let $D: P_3[\mathbb{R}] \rightarrow P_3[\mathbb{R}]$ be the linear transformation defined by

$$D(p(x)) = p'(x).$$

- Find a basis and the dimension of the kernel of D .
- Find a basis and the dimension of the image of D .

- 7) Find a linear transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ whose image is spanned by $\{[4, 1, 4], [-1, 4, 1]\}$.
- 8) Find a linear transformation $T: \mathbb{R}^4 \rightarrow \mathbb{R}^3$ whose kernel is spanned by $\{[0, 1, 1, 1], [1, 2, 3, 4]\}$.
- 9) Let $T: V \rightarrow U$ be a linear transformation. Prove that if $\dim(\text{Im}T) = \dim(\text{Ker}T)$, then the dimension of V is even.
- 10) Is it possible for a linear transformation $T: \mathbb{R}^4 \rightarrow \mathbb{R}^3$ to be one-to-one?

Answer Key

- 1) a. $B_{\text{ker}T} = \{[0, 0, 1, 4]\}$, $\dim(\text{Ker}T) = 1$
b. $B_{\text{Im}T} = \{[1, 0, 4], [0, 1, -3], [0, 0, 1]\}$, $\dim(\text{Im}T) = 3$
- 2) a. $B_{\text{ker}T} = \{(0, 0, 0)\}$, $\dim(\text{Ker}T) = 0$
b. $B_{\text{Im}T} = \{[1, 1, 0, 1], [0, 5, 1, 4], [0, 0, -6, 21]\}$, $\dim(\text{Im}T) = 3$
- 3) a. $B_{\text{ker}T} = \{[-7, 3, 0, 1], [1, -2, 1, 0]\}$, $\dim(\text{Ker}T) = 2$
b. $B_{\text{Im}T} = \{[1, 1, 2], [0, 1, 2]\}$, $\dim(\text{Im}T) = 2$
- 4) a. $B_{\text{ker}T} = \left\{ \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$, $\dim(\text{Ker}T) = 2$
b. $B_{\text{Im}T} = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \right\}$, $\dim(\text{Im}T) = 2$
- 5) a. $B_{\text{ker}T} = \{1\}$, $\dim(\text{Ker}T) = 1$ b. $B_{\text{ker}T} = \{2x + 5, 1\}$, $\dim(\text{Im}T) = 2$
- 6) a. $B_{\text{ker}D} = \{1\}$, $\dim(\text{Ker}D) = 1$ b. $B_{\text{Im}D} = \{x^2, x, 1\}$, $\dim(\text{Im}D) = 3$
- 7) $T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 & -1 & -1 \\ 1 & 4 & 4 \\ 4 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$
- 8) $T[x, y, z, t] = [-x - y + z, -2x - y + t, 0]$
- 9) Proved as shown in the videos
- 10) No, T is not one-to-one.

Isomorphism and Inverse

Questions:

- 1) Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation defined by

$$T[x, y, z] = [x - y + z, y + z, z - x].$$

True or false:

- T is one-to-one.
 - T is onto.
 - T is an isomorphism.
 - T has an inverse. If it does, find it.
- 2) Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation defined by

$$T[x, y, z] = [x - y + z, y + z, x + 2z].$$

True or false:

- T is one-to-one.
 - T is onto.
 - T is an isomorphism.
 - T has an inverse. If it does, find it.
- 3) Let $T: P_2[\mathbb{R}] \rightarrow \mathbb{R}^3$ be the linear transformation defined by

$$T(a + bx + cx^2) = [a + b + c, a - b, b - 2c].$$

True or false:

- T is one-to-one.
 - T is onto.
 - T is an isomorphism.
 - T has an inverse. If it does, find it.
- 4) Let $T: M_2[\mathbb{R}] \rightarrow P_3[\mathbb{R}]$ be the linear transformation defined by

$$T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a - b + (c + d)x + (a - c)x^2 + dx^3.$$

True or false:

- T is one-to-one.
- T is onto.
- T is an isomorphism.
- T has an inverse. If it does, find it.

Answer Key:

1) a. True

b. True

c. True

d. True, $T^{-1}[x, y, z] = \left[\frac{1}{3}(x + y - 2z), \frac{1}{3}(2y - z - x), \frac{1}{3}(z + x + y) \right]$.

2) a. False

b. False

c. False

d. False

3) a. True

b. True

c. True

d. True, $T^{-1}[a, b, c] = (0.4a + 0.6b + 0.2c)\mathbf{1} + (0.4a - 0.4b + 0.2c)x + (0.2a - 0.2b - 0.4c)x^2$

4) a. True

b. True

c. True

d. True, $T^{-1}[a, b, c, d] = [b + c - d, -a + b + c - d, b - d, d]$.

Composition of Linear Transformation

Questions:

- 1) Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ and $S: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the linear transformation defined by $T[x, y, z] = [x, 4x - y, x + 4y - z]$, $S[x, y, z] = [x - z, y]$.
Find a formula, if possible, that defines $S + T$.
- 2) Let $S: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the linear transformation defined by $S[x, y, z] = [x - z, y]$.
Find a formula, if possible, that defines $4S$.
- 3) Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ and $S: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the linear transformation defined by $T[x, y, z] = [x, 4x - y, x + 4y - z]$, $S[x, y, z] = [x - z, y]$.
Find a formula, if possible, that defines $4S - 10T$.
- 4) Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ and $S: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the linear transformation defined by $T[x, y, z] = [x, 4x - y, x + 4y - z]$, $S[x, y, z] = [x - z, y]$.
Find a formula, if possible, that defines TS , meaning function composition $T \circ S$.
- 5) Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ and $S: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the linear transformation defined by $T[x, y, z] = [x, 4x - y, x + 4y - z]$, $S[x, y, z] = [x - z, y]$.
Find a formula, if possible, that defines ST , meaning function composition $S \circ T$.
- 6) Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation defined by $T[x, y, z] = [x, 4x - y, x + 4y - z]$.
Find a formula, if possible, that defines $T^2 = TT$.
- 7) Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation defined by $T[x, y, z] = [x, 4x - y, x + 4y - z]$.
Find a formula, if possible, that defines T^{-1} .
- 8) Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation defined by $T[x, y, z] = [x, 4x - y, x + 4y - z]$.
Find a formula, if possible, that defines T^{-2} .
- 9) Let $S: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the linear transformation defined by $S[x, y, z] = [x - z, y]$.
Find a formula, if possible, that defines $S^2 = SS$.

Answer Key

- 1) $S + T$ can't be defined.
- 2) $[4(x - z), 4y]$
- 3) $4S - 10T$ can't be defined.
- 4) $T \circ S$ can't be defined.
- 5) $ST \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4y + z \\ 4x - y \end{bmatrix}$
- 6) $[x, y, 16x - 8y + z]$
- 7) $T^{-1}[x, y, z] = [x, 4x - y, 17x - 4y - z]$
- 8) $T^{-2}[x, y, z] = [x, y, -16x + 8y + z]$
- 9) $S^2 = SS$ can't be defined.