



Workbook



Table of Contents

Vector Spaces.....	3
The Vector Space \mathbb{R}^n	3
Linear Combinations and Span in \mathbb{R}^n	4
Basis for \mathbb{R}^n	6
Basis for a Solution Space, Homogeneous SLE A in \mathbb{R}^n	8
Basis of a Subspace of \mathbb{R}^n	9
Basis for Row and Column Spaces	10
Vector spaces	11
Linear Dependence	13
Basis for Known Vector Spaces	15
Basis for a Solution Space, Homogeneous SLE	16
Basis of a Subspace of a Known Vector Space	17
Coordinate Vectors and Change of Basis.....	18

Vector Spaces

The Vector Space \mathbb{R}^n

Questions

- 1) Check if W is a subspace of \mathbb{R}^3 , where $W = \{\langle a, b, c \rangle \mid a + b + c = 0\}$.
- 2) Check if W is a subspace of \mathbb{R}^3 , where $W = \{\langle a, b, c \rangle \mid a = c\}$.
- 3) Check if W is a subspace of \mathbb{R}^3 , where $W = \{\langle a, b, c \rangle \mid a = 3b\}$.
- 4) Check if W is a subspace of \mathbb{R}^3 , where $W = \{\langle a, b, c \rangle \mid a < b < c\}$.
- 5) Check if W is a subspace of \mathbb{R}^3 , where $W = \{\langle a, b, c \rangle \mid a = c^2\}$.
- 6) Check if W is a subspace of \mathbb{R}^3 , where $W = \{\langle a, b, c \rangle \mid c - b = b - a\}$.
- 7) Check if W is a subspace of \mathbb{R}^3 , where $W = \{\langle a, b, c \rangle \mid b = a \cdot q, c = a \cdot q^2\}$.

Answer Key

- 1) W is a subspace of \mathbb{R}^3 .
- 2) W is a subspace of \mathbb{R}^3 .
- 3) W is a subspace of \mathbb{R}^3 .
- 4) W is not a subspace of \mathbb{R}^3 .
- 5) W is not a subspace of \mathbb{R}^3 .
- 6) W is a subspace of \mathbb{R}^3 .
- 7) W is not a subspace of \mathbb{R}^3 .

Liner Combinations and Span in \mathbb{R}^n

Questions

1) We are given the following vectors in \mathbb{R}^4 :

$$u_1 = \langle 4, 1, 1, 5 \rangle, \quad u_2 = \langle 0, 11, -5, 3 \rangle, \quad u_3 = \langle 2, -5, 3, 1 \rangle, \quad u_4 = \langle 1, 3, -1, 2 \rangle.$$

- Is u_1 a linear combination of u_4 ?
- Does u_1 belong to $Sp\{u_4\}$?
- Is the set $\{u_1, u_4\}$ linearly dependent?

2) We are given the following vectors in \mathbb{R}^4 :

$$u_1 = \langle 4, 1, 1, 5 \rangle, \quad u_2 = \langle 0, 11, -5, 3 \rangle, \quad u_3 = \langle 2, -5, 3, 1 \rangle, \quad u_4 = \langle 1, 3, -1, 2 \rangle.$$

- Is u_3 a linear combination of u_1 and u_2 ?
- Does u_3 belong to $Sp\{u_1, u_2\}$?
- Is the set $\{u_1, u_2, u_3\}$ linearly dependent?

If so, try to write each vector in the set as a linear combination of the others.

3) We are given the following vectors in \mathbb{R}^4 :

$$u_1 = \langle 4, 1, 1, 5 \rangle, \quad u_2 = \langle 0, 11, -5, 3 \rangle, \quad u_3 = \langle 2, -5, 3, 1 \rangle, \quad u_4 = \langle 1, 3, -1, 2 \rangle.$$

- Is u_4 a linear combination of u_1 and u_2 ?
- Does u_4 belong to $Sp\{u_1, u_2\}$?
- Is the set $\{u_1, u_2, u_4\}$ linearly dependent?

If so, try to write each vector in the set as a linear combination of the others.

4) We are given the following vectors in \mathbb{R}^4 :

$$u_1 = \langle 4, 1, 1, 5 \rangle, \quad u_2 = \langle 0, 11, -5, 3 \rangle, \quad u_3 = \langle 2, -5, 3, 1 \rangle, \quad u_4 = \langle 1, 3, -1, 2 \rangle,$$

$$\text{and } v = \langle 4, 12, k, -2k \rangle.$$

- What should the value of k be, in order for the vector v to be a linear combination of u_1 and u_2 ?
- What should the value of k be, in order for the vector v to belong to $Sp\{u_1, u_2\}$?
- What should the value of k be in order for the set $\{u_1, u_2, v\}$ to be linearly dependent?

- 5) We are given the following vectors in \mathbb{R}^4 :
 $u_1 = \langle 4, 1, 1, 5 \rangle$, $u_2 = \langle 0, 11, -5, 3 \rangle$, $u_3 = \langle 2, -5, 3, 1 \rangle$, $u_4 = \langle 1, 3, -1, 2 \rangle$,
 and $v = \langle a, b, c, d \rangle$.
- What are the conditions on a, b, c, d , in order for v to be a linear combination of u_1 and u_2 ?
 - What are the conditions on a, b, c, d , in order for v to belong to $Sp\{u_1, u_2\}$?
 - What are the conditions on a, b, c, d , in order for the set $\{u_1, u_2, v\}$ to be linearly dependent?

Answer Key

- 1) a. No b. No c. No
- 2) a. Yes. $u_3 = \frac{1}{2}(u_1 - u_2)$ b. Yes. Follows from a.
 c. Yes $u_1 = 2u_3 + u_2$, $u_2 = u_1 - 2u_3$
- 3) a. Yes. $u_4 = \frac{1}{4}(u_1 + u_2)$ b. Yes. Follows from a.
 c. Yes. $u_1 = 4u_4 - u_2$, $u_2 = 4u_4 - u_1$
- 4) a-c. $k = -4$
- 5) a+b. Refer to the video. c. $\begin{cases} a = 5t + 3s \\ b = 4t - 13s \\ c = 7s \\ d = 7t \end{cases}$

Basis for \mathbb{R}^n

Questions

- 1) Check if each of the following sets is a basis of \mathbb{R}^3 :
- $\{\langle 1, 0, 1 \rangle, \langle 0, 0, 1 \rangle\}$
 - $\{\langle 1, 1, 2 \rangle, \langle 1, 2, 3 \rangle, \langle 3, 3, 4 \rangle, \langle 2, 2, 1 \rangle\}$
 - $\{\langle 1, 2, 3 \rangle, \langle 4, 5, 6 \rangle, \langle 7, 8, 9 \rangle\}$
- 2) Check if each of the following sets is a basis of $M_{2 \times 2}[\mathbb{R}]$ (AKA $M_2[\mathbb{R}]$):
- $\left\{ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}, \begin{bmatrix} 9 & 1 \\ 2 & 3 \end{bmatrix} \right\}$
 - $\left\{ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}, \begin{bmatrix} 9 & 1 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}, \begin{bmatrix} 5 & 16 \\ 7 & 8 \end{bmatrix} \right\}$
 - $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$
- 3) Check if each of the following sets is a basis of $P_2[\mathbb{R}]$ (deg ≤ 2 poly):
- $\{1+x, x^2+2x+3\}$
 - $\{1+x, x^2+2x+3, 2x+4x^2, x-x^2\}$
 - $\{1+2x+3x^2, 4+5x+6x^2, 7+8x+10x^2\}$
- 4) Consider the following set of vectors in \mathbb{R}^3 : $T = \{\langle 1, 2, 3 \rangle, \langle 4, 5, 6 \rangle, \langle 7, 8, 9 \rangle, \langle 2, 3, 4 \rangle\}$.
- Is T a basis of \mathbb{R}^3 ?
 - Find a maximal linearly independent subset T' of T .
 - Extend T' to a basis of \mathbb{R}^3 .

Answer Key

- 1)
 - a. The two vectors can't form a basis for \mathbb{R}^3 .
 - b. The four vectors can't form a basis for \mathbb{R}^3 .
 - c. The vectors are linearly dependent and so are not a basis for \mathbb{R}^3 .
- 2)
 - a. The three vectors can't form a basis for $M_{2 \times 2}[\mathbb{R}]$.
 - b. The five vectors can't form a basis for $M_{2 \times 2}[\mathbb{R}]$.
 - c. The four vectors do form a basis for $M_{2 \times 2}[\mathbb{R}]$.
- 3)
 - a. The two vectors can't form a basis for $P_2[\mathbb{R}]$.
 - b. The four vectors can't form a basis for $P_2[\mathbb{R}]$.
 - c. The three vectors do form a basis for $P_2[\mathbb{R}]$.
- 4)
 - a. No, since it contains more than three vectors.
 - b. $T' = \{\langle 1, 2, 3 \rangle, \langle 4, 5, 6 \rangle\}$
 - c. $T^* = \{\langle 1, 2, 3 \rangle, \langle 4, 5, 6 \rangle, \langle 0, 0, 1 \rangle\}$

Basis for a Solution Space, Homogeneous SLE A in \mathbb{R}^n

Questions

1) Here are 3 systems of linear equations (SLEs):

$$1. \begin{cases} x + y - z + 2w = 0 \\ 3x - y + 7z + 4w = 0 \\ -5x + 3y - 15z - 6w = 0 \end{cases} \quad 2. \begin{cases} x - y + z + w = 0 \\ x + 2z - w = 0 \\ x + y + 3z - 3w = 0 \end{cases} \quad 3. \begin{cases} x - y + z + w = 0 \\ 2x - 2y + 2z + 2w = 0 \end{cases}$$

Let's denote by W , U and V the subspaces spanned by 1., 2. and 3., respectively.

- a. Find a basis for each of W , U and V as well as their dimension.
 - b. i. Find a basis for $U + V$ and its dimension.
ii. Find the dimension of $U \cap V$.
 - c. Find a basis for $U \cap V$.
- 2) Let $U = \{ \langle a, b, c, d \rangle \in \mathbb{R}^4 \mid a = c, b = d \}$. Find a basis and the dimension of U .
- 3) Let $U = \{ \langle a, b, c, d \rangle \in \mathbb{R}^4 \mid c = a + b, d = b + c \}$. Find a basis and the dimension of U .
- 4) Let $U = \{ v \in \mathbb{R}^4 \mid v \cdot \langle 1, -1, 1, -1 \rangle = 0 \}$. Find a basis and the dimension of U .

Answer Key

- 1) a. $B_W = \{ \langle -1.5, 2.5, 1, 0 \rangle, \langle -1.5, -0.5, 0, 1 \rangle \}$ $\dim W = 2$,
 $B_U = \{ \langle -2, -1, 1, 0 \rangle, \langle 1, 2, 0, 1 \rangle \}$ $\dim U = 2$,
 $B_V = \{ \langle -1, 0, 0, 1 \rangle, \langle -1, 0, 1, 0 \rangle, \langle 1, 1, 0, 0 \rangle \}$ $\dim V = 3$
- b. i. $B_{U+V} = \{ \langle 0, 0, -1, 1 \rangle, \langle 0, 1, 1, 0 \rangle, \langle 1, 1, 0, 0 \rangle \}$ $\dim U + V = 3$
 ii. $2 = \dim(U \cap V)$
- c. $B_{U \cap V} = \{ \langle -2, -1, 1, 0 \rangle, \langle 1, 2, 0, 1 \rangle \}$
- 2) $B_U = \{ \langle 0, 1, 0, 1 \rangle, \langle 1, 0, 1, 0 \rangle \}$, $\dim U = 2$
- 3) $B_U = \{ \langle -1, 1, 0, 1 \rangle, \langle 2, -1, 1, 0 \rangle \}$, $\dim U = 2$
- 4) $B_U = \{ \langle 1, 0, 0, 1 \rangle, \langle -1, 0, 1, 0 \rangle, \langle 1, 1, 0, 0 \rangle \}$ $\dim U = 3$

Basis of a Subspace of \mathbb{R}^n

Questions

- 1) Consider the subspace of \mathbb{R}^4 defined as follows:
 $U = \text{span}\{\langle 1, 1, -1, 2 \rangle, \langle 3, -1, 7, 4 \rangle, \langle -5, 3, -15, -6 \rangle\}$.
- Find a basis for U and its dimension.
 - Find a homogeneous SLE whose solution space is U .
- 2) Consider the subspace of \mathbb{R}^4 defined as follows:
 $V = \text{span}\{\langle 1, 1, -1, 1 \rangle, \langle 1, 0, 2, -1 \rangle, \langle 1, 1, 3, -3 \rangle, \langle 5, 1, 5, 8 \rangle\}$.
 Find, for V : a basis, the dimension, a homogeneous SLE.
- 3) Consider the two subspaces of \mathbb{R}^4 defined as follows:
 $U = \text{span}\{\langle 1, 1, -1, 2 \rangle, \langle 3, -1, 7, 4 \rangle, \langle -5, 3, -15, -6 \rangle\}$
 $V = \text{span}\{\langle 1, -1, 1, 1 \rangle, \langle 1, 0, 2, -1 \rangle, \langle 1, 1, 3, -3 \rangle, \langle 5, 1, 5, 8 \rangle\}$.
 Find a basis and the dimension of $U + V$.
- 4) Consider the two subspaces of \mathbb{R}^4 defined as follows:
 $U = \text{span}\{\langle 1, 1, -1, 2 \rangle, \langle 3, -1, 7, 4 \rangle, \langle -5, 3, -15, -6 \rangle\}$
 $V = \text{span}\{\langle 1, -1, 1, 1 \rangle, \langle 1, 0, 2, -1 \rangle, \langle 1, 1, 3, -3 \rangle, \langle 5, 1, 5, 8 \rangle\}$.
 Find a basis and the dimension of $U \cap V$.

Answer Key

- 1) a. $B_U = \{\langle 1, 1, -1, 2 \rangle, \langle 0, -4, 10, -2 \rangle\}$, $\dim U = 2$ b. $\begin{cases} -3x + 5y + 2z = 0 \\ -3x - y + 2t = 0 \end{cases}$
- 2) $-8x - y + 5z + 2t = 0$, $B_V = \{\langle 1, -1, 1, 1 \rangle, \langle 0, 1, 1, -2 \rangle, \langle 0, 0, -2, 5 \rangle\}$, $\dim V = 3$
- 3) $B_{U+V} = \{\langle 1, 1, -1, 2 \rangle, \langle 0, -4, 10, -2 \rangle, \langle 0, 0, 1, 0 \rangle, \langle 0, 0, 0, 1 \rangle\}$, $\dim U + V = 4$
- 4) $\dim U \cap V = 1$, $B_{U \cap V} = \{\langle 5, 1, 5, 8 \rangle\}$

Basis for Row and Column Spaces

Questions

- 1) Find a basis and the dimension of the row space and the column space of the following matrix. What is its rank?

$$\begin{bmatrix} 4 & 1 & 1 & 5 \\ 0 & 11 & -5 & 3 \\ 2 & -5 & 3 & 1 \\ 1 & 3 & -1 & 2 \end{bmatrix}$$

- 2) Find a basis and the dimension of the row space and the column space of the following matrix. What is its rank?

$$\begin{bmatrix} 1 & 2 & 1 & 3 & 5 \\ 2 & 5 & 3 & 1 & 6 \\ 1 & -1 & -2 & 2 & 1 \\ -2 & 3 & 5 & -4 & -1 \end{bmatrix}$$

Answer Key

- 1) $B_{\text{row}} = \{\langle 4, 1, 1, 5 \rangle, \langle 0, 11, -5, 3 \rangle\}$ $\dim(\text{row}) = 2$;

$$B_{\text{col}} = \left\{ \begin{bmatrix} 4 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ -2 \\ 1 \end{bmatrix} \right\}, \dim(\text{col}) = 2;$$

rank=2.

- 2) $B_{\text{row}} = \{\langle 1, 2, 1, 3, 5 \rangle, \langle 0, 1, 1, -5, -4 \rangle, \langle 0, 0, 0, 1, 1 \rangle\}$; $\dim(\text{row}) = 3$.

$$B_{\text{col}} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -16 \\ 37 \end{bmatrix} \right\}; \dim(\text{col}) = 3; \text{Rank}=3.$$

Vector spaces

Questions

Check if W is a subspace of $M_n[\mathbb{R}]$, where:

- 1) $W = \{A \mid A = A^T\}$.
- 2) W is the set of matrices which *commute* with a given matrix B .
That is, $W = \{A \mid AB = BA\}$.
- 3) W is the set of matrices whose determinant is 0. That is, $W = \{A \mid \det A = 0\}$.
- 4) W is the set of matrices which are equal to their own square.
That is, $W = \{A \mid A^2 = A\}$.
- 5) W is the set of upper-triangular matrices.
- 6) W is the set of matrices whose product with a given matrix B is 0.
That is, $W = \{A \mid AB = \mathbf{0}\}$.
- 7) W is the set of matrices whose trace is 0.
That is, $W = \{A \mid \text{tr}(A) = 0\}$.
- 8) W is the set of matrices such that the sum of each row is 0.

Check if W is a subspace of $P_n[\mathbb{R}]$, where:

- 9) W consists of the polynomials having 4 as a root. I.e., $W = \{p(x) \mid p(4) = 0\}$.
- 10) W consists of the polynomials with degree ≤ 4 . I.e., $W = \{p(x) \mid \deg(p) \leq 4\}$.
- 11) W consists of the polynomials with integer coefficients.
- 12) W consists of the polynomials with only even powers of x in its terms.
- 13) W consists of the polynomials having degree n where $4 \leq n \leq 7$.
- 14) $W = \{p(x) \mid p(0) = 1\}$.

Check if W is a subspace of $F[\mathbb{R}]$, where:

15) W consists of all even functions. I.e., $W = \{f(x) \mid f(x) = f(-x) \text{ for all } x \in \mathbb{R}\}$.

16) W consists of all bounded functions. I.e.,
 $W = \{f(x) \mid |f(x)| \leq M \text{ for all } x \in \mathbb{R}, \text{ for some } M > 0\}$.

17) W consists of all continuous functions.

18) W consists of all differentiable functions.

19) W consists of all constant functions.

20) $W = \left\{ f(x) \mid \int_0^1 f(x) dx = 4 \text{ (assume } f \text{ is integrable)} \right\}$.

21) $W = \{f(x) \mid f'(x) = 0 \text{ (assume } f \text{ is differentiable)}\}$.

22) $W = \{f(x) \mid f'(x) = 1 \text{ (assume } f \text{ is differentiable)}\}$.

23) $W = \{f \mid f(x+1) = f(x) \text{ for all } x \in \mathbb{R}\}$.

Check if W is a subspace of $\mathbb{C}^3[\mathbb{R}]$:

24) Check if W is a subspace of $\mathbb{C}^3[\mathbb{R}]$, where $W = \{\langle z_1, z_2, z_3 \rangle \mid z_2 = \bar{z}_1, z_3 = z_1 + \bar{z}_1\}$.

25) Check if $W = \{\langle z_1, z_2, z_3 \rangle \mid z_2 = \bar{z}_1, z_3 = z_1 + \bar{z}_1\}$
 is a subspace of \mathbb{C}^3 (over the complex field \mathbb{C}).

Answer Key

1)-2) Is a subspace

3)-4) Not a subspace

5)-10) Not a subspace

11) Not a subspace

12) Is a subspace

13)-14) Not a subspace

15)-19) Is a subspace

20) Not a subspace

21) Is a subspace

22) Not a subspace

23)-24) Is a subspace

25) Not a subspace

Linear Dependence

Questions

- 1) We are given the following matrices in $M_2[\mathbb{R}]$:

$$A = \begin{bmatrix} 4 & 1 \\ 1 & 5 \end{bmatrix}, B = \begin{bmatrix} 0 & 11 \\ -5 & 3 \end{bmatrix}, C = \begin{bmatrix} 2 & -5 \\ 3 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}$$

- Are these matrices linearly dependent?
 - If so, try to write each of them as a linear combination of the rest.
 - Does A belong to $Sp\{B, C\}$?
- 2) We are given the following polynomials in $P_3[\mathbb{R}]$:
- $$p_1(x) = 4 + x + x^2 + 5x^3, p_2(x) = 11x - 5x^2 + 3x^3,$$
- $$p_3(x) = 2 - 5x + 3x^2 + x^3, p_4(x) = 1 + 3x - x^2 + 2x^3$$
- Are these polynomials linearly dependent?
 - If so, try to write each of them as a linear combination of the rest.
 - Does p_2 belong to $Sp\{p_1, p_3\}$?
- 3) We are given the following set of vectors in \mathbb{R}^3 : $S = \{\langle c, 2, 4 \rangle, \langle 2, 4, a, 2 \rangle, \langle c, b, 6 \rangle, \langle b, 2, a \rangle\}$
For which values of a, b, c is S linearly dependent?
- 4) We are given that the set $\{u, v, w\}$ of vectors is linearly independent in $V[F]$.
- Is the set $\{u - v, u - w, u + v - 2w\}$ linearly dependent?
 - If so, try to write each vector in the set as a linear combination of the others.
- 5) We are given that the set $\{u, v, w\}$ of vectors is linearly independent in $V[F]$.
- Is the set $\{u + v, v + w, w\}$ linearly dependent?
 - If so, try to write vector in the set as a linear combination of the others.
- 6) We are given that the set $\{u, v, w\}$ of vectors is linearly independent in $V[F]$.
- Is the set $\{u + 2v + 3w, 4u + 5v + 6w, 7u + 8v + 9w\}$ linearly dependent?
 - If so, try to write each vector in the set as a linear combination of the others.
- 7) Is the set of vectors $\{\langle 1, i, i - 1 \rangle, \langle i + 1, i - 1, -2 \rangle\}$ linearly independent in $\mathbb{C}^3[\mathbb{C}]$?
- 8) Is the set of vectors $\{\langle 1, i, i - 1 \rangle, \langle i + 1, i - 1, -2 \rangle\}$ linearly independent in $\mathbb{C}^3[\mathbb{R}]$?

Answer Key

- 1) a. Yes, they're linearly dependent.
b. $A = B + 2C$, $B = A - 2C$, $C = \frac{1}{2}A - \frac{1}{2}B$, $D = \frac{1}{4}A + \frac{1}{4}B$
c. Yes, follows from $A = B + 2C$.
- 2) a. Yes, they are linearly dependent.
b. $p_1 = p_2 + 2p_3$, $p_2 = p_1 - 2p_3$, $p_3 = \frac{1}{2}p_1 - \frac{1}{2}p_2$, $p_4 = \frac{1}{4}p_1 + \frac{1}{4}p_2$
c. Yes, follows from $p_2 = p_1 - 2p_3$.
- 3) For all values a, b, c S linearly is dependent.
- 4) a. Yes, they are linearly dependent.
 $x = 2y - z$
b. $y = 0.5x + 0.5z$
 $z = 2y - x$
- 5) a. No
b. N/A
- 6) a. Yes, they are linearly dependent.
 $x = 2y - z$
b. $y = 0.5x + 0.5z$
 $z = 2y - x$
- 7) No, the vectors are linearly dependent.
- 8) The vectors are linearly independent.

Basis for Known Vector Spaces

Questions

1) Check if each of the following sets is a basis of $M_{2 \times 2}[\mathbb{R}]$ (A.K.A $M_2[\mathbb{R}]$):

a. $\left\{ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}, \begin{bmatrix} 9 & 1 \\ 2 & 3 \end{bmatrix} \right\}$

b. $\left\{ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}, \begin{bmatrix} 9 & 1 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}, \begin{bmatrix} 5 & 16 \\ 7 & 8 \end{bmatrix} \right\}$

c. $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

2) Check if each of the following sets is a basis of $P_2[\mathbb{R}]$ (deg ≤ 2 poly):

a. $\{1+x, x^2+2x+3\}$

b. $\{1+x, x^2+2x+3, 2x+4x^2, x-x^2\}$

c. $\{1+2x+3x^2, 4+5x+6x^2, 7+8x+10x^2\}$

Answer Key

- 1) a. No, the three vectors can't form a basis.
 b. No, the five vectors can't form a basis.
 c. Yes, the four vectors do form a basis.
- 2) a. No, the two vectors can't form a basis.
 b. No, the four vectors can't form a basis.
 c. Yes, the three vectors do form a basis.

Basis for a Solution Space, Homogeneous SLE

Questions

1) Let $U = \{A \in M_2[\mathbb{R}] \mid A = A^T\}$. Symmetric 2x2 matrices.

Find a basis and the dimension of U .

2) Let $U = \left\{ A \in M_2[\mathbb{R}] \mid A \cdot \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$.

Find a basis and the dimension of U .

3) Let $U = \{p(x) \in P_3[\mathbb{R}] \mid p(1) = 0\}$

Find a basis and the dimension of U .

Answer Key

1) $B_U = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}, \dim U = 3.$

2) $U = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}, \dim U = 0, B_U = \emptyset$ [empty set].

3) $B_U = \{p_1(x) = -1 + x^3, p_2(x) = -1 + x^2, p_3(x) = -1 + x\}, \dim U = 3$

Basis of a Subspace of a Known Vector Space

Questions

- 1) Consider the subspace of $M_2[\mathbb{R}]$ defined as follows:

$$U = \text{span} \left\{ \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 4 & 1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ 1 & -3 \end{bmatrix} \right\}.$$

Find a basis and the dimension of U .

- 2) Consider the subspace of $P_3[\mathbb{R}]$ defined as follows:

$$U = \text{span} \{1 + x - x^2 + 2x^3, 4 + x - x^2 + x^3, 2 - x + x^2 - 3x^3\}$$

Find a basis and the dimension of U .

Answer Key

$$1) \quad B_U = \left\{ \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & -3 \\ 3 & -7 \end{bmatrix} \right\}, \quad \dim U = 2$$

$$2) \quad B_U = \{1 + x - x^2 + 2x^3, -3x + 3x^2 - 7x^3\}, \quad \dim U = 2$$

Coordinate Vectors and Change of Basis

Questions

- 1) Given the following two bases of $P_2[\mathbb{R}]$:

$$B_1 = \{1+x, x, x+x^2\}; \text{ and } B_2 = \{1+x^2, x+x^2, x^2\},$$

and let $p(x) = a+bx+cx^2$, be a general polynomial in $P_2[\mathbb{R}]$.

Compute $[p(x)]_{B_1}$, the coordinate vector of $p(x)$ relative to B_1 and B_2 .

Find the change-of-basis matrix from B_1 to B_2 .

- 2) Given the following two bases of $M_2[\mathbb{R}]$:

$$B = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$E = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Compute the coordinate vector of $X = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$, relative to B and E .

Find the change-of-basis matrix from B to E .

Answer Key

$$1) [v]_{B_1} = \langle a, b-a-c, c \rangle; \quad [v]_{B_2} = \langle a, b, c-a-b \rangle, \quad [M]_{B_1}^{B_2} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$2) [X]_B = \langle x, y-x, z-y+x, t-z+y-x \rangle$$

E is the elementary or standard basis of $M_2[\mathbb{R}]$:

$$[X]_E = \langle x, y, z, t \rangle$$

The change-of-basis matrix from B to E : $[M]_B^E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{bmatrix}$