proprep™

Workbook

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Linear Transformations

Linear Transformation Definition

Questions:

Check if the following transformations are linear transformations:

1) $T:\mathbb{R}^2 \to \mathbb{R}^2$; $T[x,y]=[x+y,x-y]$

2) $T:\mathbb{R}^3 \to \mathbb{R}^3$; $T[x,y,z]=[x+y-2z,x+2y+z,2x+2y-3z]$

3) $T:\mathbb{R}^3 \to \mathbb{R}^2$; $T[x,y,z]=[2x+z,|y|]$

4) $T:\mathbb{R}^2 \to \mathbb{R}^3$; $T[x,y]=[xy,y,z]$

5) $T:\mathbb{R}^3 \to \mathbb{R}^3$; $T[x,y,z]=[x+1,x+y,x+z]$

6) $T:M_n[\mathbb{R}]\to M_n[\mathbb{R}]$; $T(A)=BA+AB$, where $B\in M_n[R]$ is some fixed matrix.

7) $T:M_n[\mathbb{R}]\to M_n[\mathbb{R}]$; $T(A)=A+A^T$

8) $T:M_n[\mathbb{R}]\to M_n[\mathbb{R}]$; $T(A)=|A|\cdot I$ ($|A|$ is the determinant of A)

9) $T:M_n[\mathbb{R}]\to M_n[\mathbb{R}]$; $T(A)=A\cdot A^T$

10) $T:M_n[\mathbb{R}]\to M_n[\mathbb{R}]$; $T(A)=A^3$

11) $T:P_3[\mathbb{R}]\to P_2[\mathbb{R}]$; $T\left(a+bx+cx^2+dx^3\right)=a+bx+cx^2$

12) $T:P_n[\mathbb{R}]\to P_n[\mathbb{R}]$; $T\left(p(x)\right)=p(x+1)$

13) $T:P_n[\mathbb{R}]\to P_n[\mathbb{R}]$; $T\left(p(x)\right)=p'(x)+p''(x)$

14) $T:P_n[\mathbb{R}]\to P_{2n}[\mathbb{R}]$; $T\left(p(x)\right)=p^2(x)$

15) Check if the following transformations are linear transformations:

a. $T:\mathbb{C}[\mathbb{R}]\to\mathbb{C}[\mathbb{R}]\ ;\ T(z)=\overline{z}$

b. $T:\mathbb{C}[\mathbb{C}]\to\mathbb{C}[\mathbb{C}]\ ;\ T(z)=\overline{z}$

16) For which value(s) of m is the following a linear transformation?

$T:\mathbb{R}^2\to\mathbb{R}^2\ ;\ T[x,y]=\left[m^2x^{2m},y^{2m}+x\right]$

17) Is there a linear transformation $T:\mathbb{R}^4\to\mathbb{R}^3$, such that: $\begin{matrix} T[1,2,-1,0]=[0,1,-1], \\ T[-1,0,1,1]=[1,0,0], \\ T[0,4,0,2]=[2,2,-2] \end{matrix}$?

If there isn't, explain why.
If there is, find such a T and say whether this T is unique.

18) Is there a linear transformation $T:P_2[\mathbb{R}]\to P_2[\mathbb{R}]$ such that: $\begin{matrix} T(1)=4, \\ T(4x+x^2)=x, \\ T(1-x)=x^2+1 \end{matrix}$?

If there isn't, explain why.
If there is, is it unique? If T is unique, find its formula.

19) Is there a linear transformation $T:\mathbb{R}^3\to\mathbb{R}^3$, such that: $\begin{matrix} T[1,1,0]=[1,2,3] \\ T[0,1,1]=[4,5,6] \\ T[0,0,1]=[7,8,9] \end{matrix}$?

If there isn't, explain why.
If there is, is it unique? If T is unique, find its formula.

20) Is there a linear transformation $T:\mathbb{R}^3\to\mathbb{R}^3$, such that: $\begin{matrix} T[1,0,1]=[1,1,0] \\ T[0,1,1]=[1,2,1] \\ T[0,0,1]=[0,1,1] \end{matrix}$?

If there isn't, explain why.
If there is, is it unique? If T is unique, find its formula.

Answer Key:

1) T is linear.

2) T is linear.

3) T is not linear.

4) T is not linear.

5) T is not linear.

6) T is linear.

7) T is linear.

8) T is not linear.

9) T is not linear.

10) ....

11) T is linear.

12) T is linear.

13) T is linear.

14) T is not linear.

15) a. T is linear.

b. T is not linear.

16) $m = 2$

17) Yes, not unique $T[x,y,z,t] = [\frac{1}{2}y - x, \frac{1}{2}y, -\frac{1}{2}y]$.

18) Yes, unique T $\overline{T}[a,b,c] = [4a + 3b - 12c, c, 4c - b]$.

19) Yes, unique T $T[x,y,z] = [4x - 3y + 7z, 5x - 3y + 8z, 6x - 3y + 9z]$.

20) Yes, unique T $T[x,y,z] = [x + y, y + z, z - x]$.

Image and kernel

Questions:

1) Let $T:\mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation defined by $T[x,y,z,t]=[x+y,y-4z+t,4x+y+4z-t]$.

a. Find a basis and the dimension of the kernel of T.
b. Find a basis and the dimension of the image of T.

2) Let $T:\mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation defined by $T[x,y,z]=[x-4y-z,x+y,y-z,x+4z]$.

a. Find a basis and the dimension of the kernel of T.
b. Find a basis and the dimension of the image of T.

3) Let $T:\mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation defined by

$$T[x,y,z,t]=\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & -2 \\ 2 & 6 & 10 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}.$$

a. Find a basis and the dimension of the kernel of T.
b. Find a basis and the dimension of the image of T.

4) Let $T:M_2[\mathbb{R}] \to M_2[\mathbb{R}]$ be the linear transformation defined by

$$T(A)=A\cdot\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}-\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\cdot A.$$

a. Find a basis and the dimension of the kernel of T.
b. Find a basis and the dimension of the image of T.

5) Let $T:P_2[\mathbb{R}] \to P_2[\mathbb{R}]$ be the linear transformation defined by $T(p(x))=p(x+1)-p(x+4)$.

a. Find a basis and the dimension of the kernel of T.
b. Find a basis and the dimension of the image of T.

6) Let $D:P_3[\mathbb{R}] \to P_3[\mathbb{R}]$ be the linear transformation defined by $D(p(x))=p'(x)$.

a. Find a basis and the dimension of the kernel of D.
b. Find a basis and the dimension of the image of D.

7) Find a linear transformation $T:\mathbb{R}^3 \to \mathbb{R}^3$ whose image is spanned by $\{[4,1,4],[-1,4,1]\}$.

8) Find a linear transformation $T:\mathbb{R}^4 \to \mathbb{R}^3$ whose kernel is spanned by $\{[0,1,1,1],[1,2,3,4]\}$.

9) Let $T:V \to U$ be a linear transformation.
Prove that if $\dim(\operatorname{Im}T)=\dim(KerT)$, then the dimension of V is even.

10) Is it possible for a linear transformation $T:\mathbb{R}^4 \to \mathbb{R}^3$ to be one-to-one?

Answer Key :

1) a. $B_{\ker T}=\{[0,0,1,4]\}$, $\dim(KerT)=1$

b. $B_{\operatorname{Im}T}=\{[1,0,4],[0,1,-3],[0,0,1]\}$, $\dim(\operatorname{Im}T)=3$

2) a. $B_{KerT}=\{(0,0,0)\}$, $\dim(KerT)=0$

b. $B_{\operatorname{Im}T}=\{[1,1,0,1],[0,5,1,4],[0,0,-6,21]\}$, $\dim(\operatorname{Im}T)=3$

3) a. $B_{KerT}=\{[-7,3,0,1],[1,-2,1,0]\}$, $\dim(KerT)=2$

b. $B_{\operatorname{Im}T}=\{[1,1,2],[0,1,2]\}$, $\dim(\operatorname{Im}T)=2$

4) a. $B_{KerT}=\left\{\begin{bmatrix}-1 & 1\\ 0 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right\}$, $\dim(KerT)=2$

b. $B_{\operatorname{Im}T}=\left\{\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\ 1 & -1\end{bmatrix}\right\}$, $\dim(\operatorname{Im}T)=2$

5) a. $B_{KerT}=\{1\}$, $\dim(KerT)=1$ b. $B_{KerT}=\{2x+5,1\}$, $\dim(\operatorname{Im}T)=2$

6) a. $B_{KerD}=\{1\}$, $\dim(KerD)=1$ b. $B_{\operatorname{Im}D}=\{x^2,x,1\}$, $\dim(\operatorname{Im}D)=3$

7) $T\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}4 & -1 & -1\\ 1 & 4 & 4\\ 4 & 1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}$

8) $T[x,y,z,t]=[-x-y+z,-2x-y+t,0]$

9) Proved as shown in the videos

10) No, T is not one-to-one.

Isomorphism and Inverse

Questions:

1) Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by $T[x,y,z]=[x-y+z, y+z, z-x]$. True or false:

a. T is one-to-one.
b. T is onto.
c. T is an isomorphism.
d. T has an inverse. If it does, find it.

2) Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by $T[x,y,z]=[x-y+z, y+z, x+2z]$. True or false:

a. T is one-to-one.
b. T is onto.
c. T is an isomorphism.
d. T has an inverse. If it does, find it.

3) Let $T:P_2[\mathbb{R}] \to \mathbb{R}^3$ be the linear transformation defined by $T(a+bx+cx^2)=[a+b+c, a-b, b-2c]$. True or false:

a. T is one-to-one.
b. T is onto.
c. T is an isomorphism.
d. T has an inverse. If it does, find it.

4) Let $T:M_2[\mathbb{R}] \to P_3[\mathbb{R}]$ be the linear transformation defined by $T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a-b+(c+d)x+(a-c)x^2+dx^3$. True or false:

a. T is one-to-one.
b. T is onto.
c. T is an isomorphism.
d. T has an inverse. If it does, find it.

Answer Key:

1) a. True b. True c. True

d. True $T^{-1}[x,y,z]=\left[\frac{1}{3}(x+y-2z),\frac{1}{3}(2y-z-x),\frac{1}{3}(z+x+y)\right]$.

2) a. False b. False c. False d. False

3) a. True b. True c. True

d. True $T^{-1}[a,b,c]=(0.4a+0.6b+0.2c)\mathbf{1}+(0.4a-0.4b+0.2c)\boldsymbol{x}+(0.2a-0.2b-0.4c)\boldsymbol{x}^2$

4) a. True b. True c. True

d. True $T^{-1}[a,b,c,d]=[b+c-d,-a+b+c-d,b-d,d]$.

Composition of Linear Transformation

Questions:

1) Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ and $S:\mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by
$T[x,y,z]=[x,4x-y,x+4y-z]$, $S[x,y,z]=[x-z,y]$.
Find a formula, if possible, that defines $S+T$.

2) Let $S:\mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by $S[x,y,z]=[x-z,y]$.
Find a formula, if possible, that defines $4S$.

3) Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ and $S:\mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by
$T[x,y,z]=[x,4x-y,x+4y-z]$, $S[x,y,z]=[x-z,y]$
Find a formula, if possible, that defines $4S-10T$.

4) Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ and $S:\mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by
$T[x,y,z]=[x,4x-y,x+4y-z]$, $S[x,y,z]=[x-z,y]$.
Find a formula, if possible, that defines TS, meaning function composition $T \circ S$.

5) Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ and $S:\mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by
$T[x,y,z]=[x,4x-y,x+4y-z]$, $S[x,y,z]=[x-z,y]$.
Find a formula, if possible, that defines ST, meaning function composition $S \circ T$.

6) Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by
$T[x,y,z]=[x,4x-y,x+4y-z]$.
Find a formula, if possible, that defines $T^2=TT$.

7) Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by
$T[x,y,z]=[x,4x-y,x+4y-z]$.
Find a formula, if possible, that defines T^{-1}.

8) Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by
$T[x,y,z]=[x,4x-y,x+4y-z]$.
Find a formula, if possible, that defines T^{-2}.

9) Let $S:\mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by
$S[x,y,z]=[x-z,y]$.
Find a formula, if possible, that defines $S^2=SS$.

Answer Key:

1) $S+T$ can't be defined.

2) $\left[4(x-z),4y\right]$

3) $4S-10T$ can't be defined.

4) $T\circ S$ can't be defined.

5) $ST\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} -4y+z \\ 4x-y \end{bmatrix}$

6) $[x,y,16x-8y+z]$

7) ....

8) $T^{-2}[x,y,z]=[x,y,-16x+8y+z]$

9) $S^2=SS$ can't be defined.