Logarithms, Logarithmic Equations, Applications

Rules of Logarithms

Questions:

1) Compute the value of the following logarithms:

a. $\log_5 5$ b. $\log_{125} 5$ c. $\log_{\frac{1}{2}} 16$

2) Compute the value of the following logarithms:

a. $\log_{\frac{1}{4}} \frac{1}{8}$ b. $\log_{\frac{1}{\sqrt[3]{3}}} \frac{1}{9}$ c. $\log_{\frac{1}{27}} \sqrt[4]{3}$

3) Compute the value of the following logarithms:

a. $\log_{\frac{1}{\sqrt[3]{25}}} \sqrt[5]{125}$ b. $\log_{0.01} \frac{10}{\sqrt[4]{1000}}$

4) Compute the value of the following expressions:

a. $\log_2 10 + \log_2 6.4$ b. $\log_2 768 - \log_2 6$ c. $\log_{0.25} 80 - \log_{0.25} 5$

5) Compute the value of the following expressions:

a. $3\log_3 6 + \log_3 3.375$ b. $\log_5 50 + \log_5 20 - \log_5 2 - \log_5 4$

6) Compute the value of the following expressions:

a. $-\frac{1}{4}\log_{\sqrt{7}} 81 + 2\log_{\sqrt{7}} 6 - \log_{\sqrt{7}} 84$ b. $\frac{1}{2}\log_{\sqrt[3]{2}} 6 - \frac{1}{2}\log_{\sqrt[3]{2}} 3 + \frac{3}{2}\log_{\sqrt[3]{2}} 4$

7) Compute the value of the following expressions:

a. $\frac{\log_3 16}{\log_3 8}$ b. $\frac{\log_3 6 + 2}{\log_3 108 - \log_3 2}$

8) Compute the value of the expression: $\dfrac{2-3\log 5+\log 50}{1+\log 128-5\log 2}$.

Hint: convert whole numbers to logarithmic expressions using $n=\log_a a^n$.

9) Compute the value of the following expressions (the logarithm is base 10):

a. $\dfrac{\log 27}{\log 9}$ b. $\dfrac{\log 24-\log 3}{\log 2}$ c. $\dfrac{1+\log 5}{\log 2+2\log 5}$

10) Prove that the following equality holds (the logarithm is base 10):

$$\frac{\log 9+2\log 5+\log 4}{\log 10-\log 2+\log 6}=2$$

11) Given: $\log_2 7=a$. Express the following in terms of a:

a. $\log_2 14$ b. $\log_2 49$

12) Given: $\log 4=a$. Express the following in terms of a:

a. $\log 16$ b. $\log 2$ c. $\log 8$

13) Given: $\log_3 6=a$, $\log_3 5=b$. Express the following in terms of a and b:

a. $\log_3 30$ b. $\log_3 1.2$ c. $\log_3 150$

14) Compute the values of the following expressions, using the formula $a^{\log_a b}=b$:

a. $2^{\log_2 3}$ b. $10^{\log 2}$ c. $3^{3\log_3 4}$

15) Compute the values of the following expressions, using the formula $a^{\log_a b}=b$:

a. $8^{\log_2 3}$ b. $\sqrt{6}^{\log_{36} 4}$ c. $\sqrt[3]{5}^{\log_5 64}$

16) Compute the values of the following expressions, using the formula $a^{\log_a b}=b$:

a. $5^{1+\log_5 2}$ b. $\sqrt[3]{27^{1+\log_3 2}}$

17) Compute the following expressions, using the formula $\log_a b=\dfrac{\log_m b}{\log_m a}$:

a. $\log_3 6\cdot\log_6 3$ b. $\log_{0.1} 5\cdot\log_{25} 100$

18) Compute the following expression, using the formula $\log_a b = \frac{\log_m b}{\log_m a}$:

$\log_{81} 49 \cdot \log_{32} 3 \cdot \log_7 2$

19) Prove the following equality, using the formula $\log_a b = \frac{\log_m b}{\log_m a}$:

$\log_3 5 \cdot \log_5 8 + \log_3 2 \cdot \log_2 5 = \log_3 40$

20) Given: $\log_2 5 = a$. Express the following in terms of a:

a. $\log_5 2$ b. $\log_4 5$ c. $\log_{16} 5$

21) Given: $\log 2 = a$. Express the following in terms of a:

a. $\log 80$ b. $\log_8 40$ c. $\log_{80} 2000$

Answer Key:

1) a. $x=1$ b. $x=\frac{1}{3}$ c. $x=-4$

2) a. $x=\frac{3}{2}$ b. $x=6$ c. $x=-\frac{1}{2}$

3) a. $x=-\frac{9}{10}$ b. $x=-\frac{1}{8}$

4) a. $x=6$ b. $x=7$ c. $x=-2$

5) a. $x=6$ b. $x=3$

6) a. $x=-2$ b. $8\sqrt{2}$

7) a. $\frac{4}{3}$ b. 1

8) 1

9) a. 1.5 b. 3 c. 1

10) Solution in the recording.

11) a. $a+1$ b. $2a$

12) a. $2a$ b. $\frac{1}{2}a$ c. $\frac{3}{2}a$

13) a. $a+b$ b. $a-b$ c. $a+2b$

14) a. 3 b. 2 c. 64

15) a. 27 b. $\sqrt{2}$ c. 4

16) a. 10 b. $\sqrt{216}$

17) a. 1 b. -1

18) $\frac{1}{10}$

19) Solution in the recording.

20) a. $\frac{1}{a}$ b. $\frac{2}{a}$ c. $\frac{a}{4}$

21) a. $3a+1$ b. $\frac{2a+1}{3a}$ c. $\frac{a+3}{3a+1}$

Exponential Equations - Revisited

Questions:

1) Solve the following equations:

a. $2^x = 5$

b. $3^x = 7$

2) Solve the following equations:

a. $2^{x+1} = 5$

b. $3 \cdot 4^{2x+1} = 15$

Answer Key:

1) a. $x = \log_2 5$ b. $x = \log_3 7$

2) a. $x = \log_2 5 - 1$ b. $x = \log_4 5$

Logarithmic Equations - Type 1

Questions:

1) Solve the following equations:

a. $\log_x(2x-4)=1$

b. $\log_x(x^2-3x+15)=2$

2) Solve the following equations:

a. $\log_{\frac{1}{4}}(x^2-4x+11)=-\frac{3}{2}$

b. $\log_{\sqrt{27}}(x^2-8x)=\frac{4}{3}$

3) Solve the following equations:

a. $\log_{\frac{1}{2}}\left(\frac{1}{x^2-4x+11}\right)=3$

b. $\log_x\sqrt{2x+1}=-\frac{1}{2}$

4) Solve the following equations:

a. $\log_{\frac{1}{2}}(\log_{x-2}x)=-1$

b. $\log_{\sqrt{2}}(\log_{x-1}(-5+3x))=2$

5) Solve the following equations:

a. $\log_2(\log_2(x+1))^2=4$

b. $\log_3^2(x+1)=4$

6) Solve the following equations:

a. $\log_7(4+2\log_9 x)=0$

b. $\log_{\frac{1}{2}}(\log_2(2\log_3 x))=-1$

7) Solve the following equations:

a. $\log_{\frac{1}{2}}(\log_2(\log_{\sqrt{5}}(10x^2+14x)))=-1$

b. $\log_2(\log_{\sqrt{2}}(\log_2(\log_{\sqrt{3}}(2x^2+x-1))))=1$

8) Solve the following equations:

a. $\ln x=4$

b. $2\ln(4x+1)=0$

9) Solve the following equations:

a. $\ln(\ln(x))=1$

b. $\ln(\ln(x+e-1))=0$

10) Solve the following equations:

a. $\log_2\left(3^x-17\right)=6$

b. $\log_2\left(12-2^{-x}\right)=1-x$

11) Solve the following equations:

a. $3x=-3+\log_2\left(12+10\cdot 2^{3x-1}\right)$

b. $2x+3=\log_5\left(30-25^{x+1}\right)$

12) Solve the following equations:

a. $x=\frac{1}{2}\log_5\left(6\cdot 5^x-5\right)$

b. $x=\log_2\left(68-4^{x-1}\right)+2$

13) Solve the following equations:

a. $\log_3 x=2-\frac{2}{3}\log_3 64$

b. $\log_5\left(125^x-24\cdot 25^x\right)-2=x$

14) Solve the following equations:

a. $4^{\log_4(2x+5)}=3x-4$

b. $5^{\log_5\left(x^2+4x\right)}=12$

15) Solve the following equations:

a. $3^{2\log_3(x+1)}=4$

b. $4^{\frac{1}{\log_4(x+2)}}=\frac{1}{2x+5}$

Answer Key:

1) a. $x=4$ b. $x=5$

2) a. $x=1,3$ b. $x=-1,9$

3) a. $x=3,1$ b. $x=\frac{1}{2}$

4) a. $x=4,1$ b. $x=3$

5) a. $x=15,-\frac{15}{16}$ b. $x=8,-\frac{8}{9}$

6) a. $x=\frac{1}{27}$ b. $x=9$

7) a. $x=1,-2\frac{3}{11}$ b. $x=2,-2\frac{1}{2}$

8) a. $x=e^4$ b. $x=0$

9) a. $x=e^e$ b. $x=1$

10) a. $x=4$ b. $x=-2$

11) a. No solution. b. $x=-\frac{1}{2}$

12) a. $x=1,0$ b. $x=4$

13) a. $x=\frac{9}{16}$ b. $x=2$

14) a. $x=9$ b. $x=2,-6$

15) a. $x=1$ b. No solution.

Logarithmic Equations - Type 2

Questions:

1) Solve the following equations:

a. $\log x + \log 2x = \log 36 - \log 2$

b. $\log x^3 + \log \frac{1}{x} = \log 3 + \log 8 - \log 6$

2) Solve the following equations:

a. $2\log_5 (x-2) - \log_5 x = \log_5 (x-3)$

b. $\log_{\frac{1}{4}} x + 2\log_{\frac{1}{4}} \left(\sqrt{x}+3\right) = \log_{\frac{1}{4}} 16$

3) Solve the following equations:

a. $2\log_2 \sqrt{x} - \log_2 \frac{1}{x+1} = -1 + \log_2 4$

b. $\log(x+2) = \log 3 - \frac{1}{2}\log x^2$

4) Solve the following equations:

a. $\ln 2x^2 + \ln \frac{1}{x} = 2\ln \sqrt{e}$

b. $\ln 4 + \ln \frac{x}{4} + \ln 2 = \ln x + \ln(x+1)$

5) Solve the following equations:

a. $\log_2 x + 2\log_4 x + 3\log_8 x = \log_2 8$

b. $\log_5 x + \frac{1}{2}\log_{\sqrt{5}} x + 2\log_{25} x = 1.5\log_{125} x^2$

Answer Key:

1) a. $x = 3$ b. $x = 2$

2) a. $x = 4$ b. $x = 1$

3) a. $x = 1$ b. $x = 1$

4) a. $x = \frac{e}{2}$ b. $x = 1$

5) a. $x = 2$ b. $x = 1$

Logarithmic Equations - Type 3

Questions:

1) Solve the following equations:

a. $2(\log x)^2+3\log x-5=0$

b. $4\log^2 x+\log x^3-10\log 10=0$

2) Solve the following equations:

a. $2\log\sqrt{x}+2=\dfrac{8}{\log x}$

b. $1+\dfrac{2}{3\log\sqrt[3]{x}}=\dfrac{3}{\log^2 x}$

3) Solve the following equations:

a. $\dfrac{2}{\ln x-1}+\dfrac{4}{2-\ln x^{-1}}=3$

b. $2\log_2\dfrac{1}{\sqrt{x}}+(\log_4 x)^2=-\dfrac{3}{4}$

4) Solve the following equations:

a. $\log(x-6)=\log 8-3\log\sqrt[3]{x-4}$

b. $\log_x 2+\log_{2x}4=2\log_{4x}8$

Answer Key:

1) a. $x=10,10^{-2.5}$ b. $x=17.87,0.01$

2) a. $x=100,0.0001$ b. $x=10,0.001$

3) a. $x=e^2,e^{-1}$ b. $x=8,2$

4) a. $x=8$ b. $x=\dfrac{1}{2^{\frac{2}{3}}},2$

Logarithmic Equations - Type 4

Questions:

1) Solve the following equations:

a. $x^{\log_4 x} = 4$

b. $x^{\log_2 x} = 16$

2) Solve the following equations:

a. $x^{-1-\log x} = \frac{1}{100}$

b. $(3x)^{2+2\log_3 x} = 9$

3) Solve the following equations:

a. $\left(x^2\right)^{\log_4 \frac{1}{x}} = \frac{1}{64x}$

b. $x^{2\log x} = 1000000x^4$

4) Solve the following equations:

a. $0.1^{3-\log(10x)} = \frac{100}{x^3}$

b. $x^{2-\log_5 \frac{x}{5}} = \frac{1}{625}$

5) Solve the following equations:

a. $(10x)^{\frac{2}{\log x}} = 10000x^4$

b. $\left(\sqrt{x}\right)^{7+\log_3 x} = 9^{1+\log_3 x}$

6) Solve the following equations:

a. $x^{2-\log x^3} = 10\left(\frac{1}{x}\right)^{1+\log x}$

b. $\left(\sqrt{3x}\right)^{-\log_3(9x)} = x^{-3}$

7) Solve the following equations:

a. $\left(10x^2\right)^{\log x-3} = \frac{1}{x^6}$

b. $\left(4x^2\right)^{\log_x 4} = x^8$

8) Solve the following equations:

a. $625^{\log_5 x} = 4x^6$

b. $\left(\frac{x^2}{1000}\right)^{-\log x} = 0.01$

Answer Key:

1) a. $x=4,\frac{1}{4}$ b. $x=16,\frac{1}{16}$

2) a. $x=10,\frac{1}{100}$ b. $x=1,\frac{1}{9}$

3) a. $x=8,\frac{1}{4}$ b. $=1000,\frac{1}{10}$

4) a. $x=10$ b. $x=625,\frac{1}{5}$

5) a. $x=\sqrt{10},\frac{1}{10}$ b. $x=3,\frac{1}{81}$

6) a. $x=10,\sqrt{10}$ b. $x=9,3$

7) a. $x=10,\frac{1}{10\sqrt{10}}$ b. $x=2,\frac{1}{\sqrt{2}}$

8) a. $x=\frac{1}{2}$ b. $x=100,\frac{1}{\sqrt{10}}$

Applications - Decay and Growth - Version 1

Questions:

1) A woman deposited $\$500$ in a savings account at an interest rate of 3% compounded annually. Determine how much money will be in the account after:
   a. 5 years.
   b. 2 years.
   c. 4 months (Assume all months are exactly 1/12 of a year).
   d. 6 months.

2) Assume that the population of Earth is growing exponentially at a (constant) rate of 2% per year, and that in 1980 it was 4 billion (4,000,000,000).
   a. What will the population on Earth be in 2020?
   b. What was the population on Earth in 1974?
   c. When a population of 50 billion will be reached?

3) The population in a certain city grows exponentially. In a certain year there were 400 thousand residents and 4 years later there were 440 thousand.
   a. Find the annual growth rate (as a %).
   b. After how many years (from that certain year) were there 550 thousand residents?

4) A man deposited money in the bank at an interest rate of 4% compounded annually. After 5 years he had accumulated $\$5000$.
   a. How much did he initially deposit?
   b. After how many years will he have accumulated $\$7000$?

5) The number of wild animals at a nature reserve grows exponentially.
There were 1000 animals at the initial count. At a second count, 20 months later, there were 1400 wild animals.
How many months after the initial count will the reserve have 2000 animals?

6) The radioactive isotope carbon-14 decays exponentially with a half-life of 5750 years.
   a. How many grams of this isotope will remain after 1000 years, if there were 100 grams initially?
   b. After how many years will there remain just 10 grams of the initial 100 grams?

7) In a certain pool there are 240 tons of fish, and the quantity of fish in it increases by 4% each week. In a second pool there are 200 tons of fish, and the quantity of fish in it increases by 10% each week.
   a. After how many weeks both pools will have the same quantity of fish?
   b. After how many weeks will the second pool have twice the quantity of fish as the first pool?

8) John purchased a car. Assume its value depreciates exponentially. After 4 years its value was $\$5,000$. After an additional 2 years its value dropped to $\$4,000$. What was the initial purchase price?

9) The value of a share (of stock) is increasing exponentially and doubles itself in 4 years. How long does it take for the share to triple its value?

10) Assume the value of a car depreciates exponentially. After 4 years it lost 25% of its value. How many years does it take for the car to lose 50% of its value?

Answer Key:

1) a. $\$579.64$ b. $\$530.45$ c. $\$504.65$ d. $\$507.44$

2) a. 8,832,158,654 b. 3,551,885,528 c. 2107 or 2108

3) a. 2.41% b. 13.36 years

4) a. $\$4,109.63$ b. 13.5789 years

5) 41.2 months

6) a. 88.64 grams b. 19,101 years

7) a. 3.25 weeks b. 15.61 weeks

8) $\$7,812$

9) 6.34 years

10) 9.64 years

Applications - Decay and Growth - Version 2

Questions:

1) $\$10{,}000$ are deposited in an account that earns interest at an annual rate of 4%.
Determine how much money will be in the account after 26 months, if the interest is compounded:
a. Quarterly. b. Monthly. c. Continuously.

2) A bank account is opened with an initial deposit of $\$4000$. The annual interest rate is 2%.
How long will it take for the money to double if the interest is compounded:
a. Quarterly. b. Monthly. c. Continuously.

3) We deposit $\$4500$ into an account that earns interest at an annual rate of 8%.
How long do we have to wait till there are $\$7000$ in the account if the interest is compounded:
a. Continuously. b. 6 times a year.

4) The growth of a colony of bacteria is given by the equation: $Q_t = Q_0 e^{0.215t}$;
where t is in hours. Given that there are initially 400 bacteria present.
a. How many bacteria will there be after two days?
b. How long will it take till there are 20,000 bacteria in the colony?

5) A population of bacteria grows exponentially. Initially, there are 400 bacteria present and in 4 days there will be 1,200.
a. Determine the growth equation for this population (time is measured in days).
b. How long will it take for the population to reach 5,000?

6) The number of wild animals at a nature reserve grows exponentially.
There were 1000 animals at the initial count. At a second count, 20 months later, there were 1400 wild animals.
How many months after the initial count will the reserve have 2000 animals?

7) The radioactive isotope carbon-14 decays exponentially with a half-life of 5750 years
a. How many grams of this isotope will remain after 1000 years, if there were 100 grams initially?
b. After how many years will there remain just 10 grams of the initial 100 grams?

8) We initially have 200 grams of a radioactive element and in 950 years 90 grams will remain.
a. Determine the exponential decay equation for this element.
b. How long will it take for half of the element to decay?
c. How long will it take until there is only 1 gram of the element left?

Answer Key:

1) a. $\$10,900.63$ b. $\$10,903.76$ c. $\$10,905.33$

2) a. 34.74 years. b. 34.68 years. c. 34.65 years.

3) a. 5.52 years. b. 5.56 years.

4) a. 12,133,303 b. 18.195 hours.

5) a. $Q_t = 400 \cdot 3^{\frac{t}{4}}$ b. 9.196 days.

6) 41.2 months.

7) a. 88.64 grams. b. 19,101 years.

8) a. $Q_t = 200e^{\frac{t \ln 0.45}{950}}$ b. 824.65 years. c. 6,303.51 years.