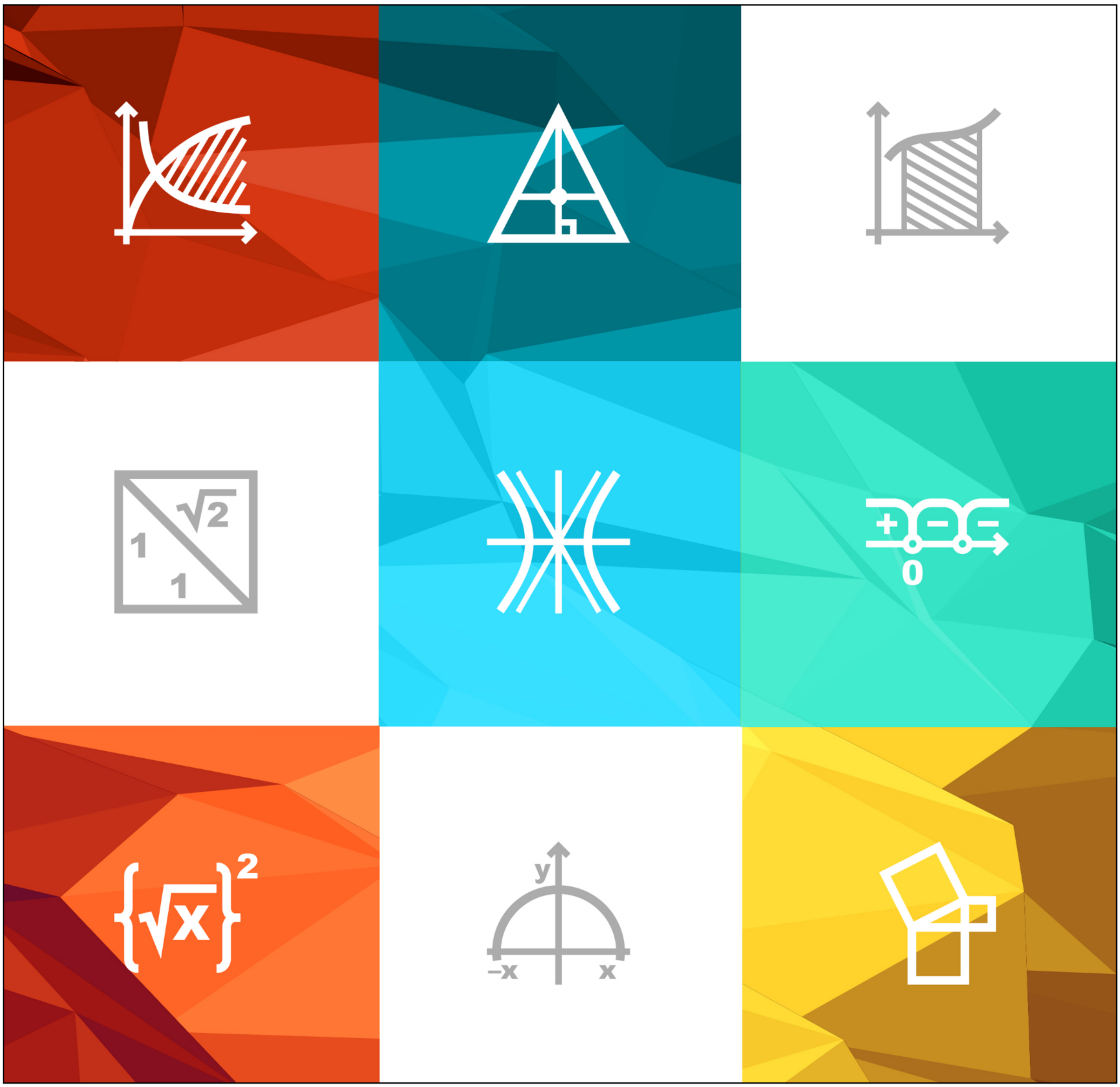
proprep™

Workbook



Tale of Contents

Matrices

Matrix and Basic Operations on Matrices

Questions:

1) For each of the following matrix "calculations", say whether it's undefined or give the order [size] of the result:

a. $A_{4\times6}+B_{4\times6}$

b. $A_{4\times6}\cdot B_{4\times6}$

c. $A_{4\times6}\cdot C_{6\times2}-D_{4\times2}$

d. $A_{4\times6}\cdot E_{6\times4}-B_{4\times6}$

e. $B_{4\times6}+A_{4\times6}\cdot B_{4\times6}$

f. $E_{6\times4}\left(B_{4\times6}+A_{4\times6}\right)$

g. $\left(E_{6\times4}+A^T_{4\times6}\right)D_{4\times2}$

h. $E^T_{6\times4}\cdot B_{4\times6}$

i. $E_{6\times4}\cdot A_{4\times6}\cdot C_{6\times2}$

j. $E_{6\times4}\left(B_{4\times6}-A_{4\times6}\right)$

2) Solve the matrix equation $\begin{bmatrix} x+2y & 3x-2y \\ 2x-5y & 2x+8y \end{bmatrix}=\begin{bmatrix} 2-2z & 5+z \\ -4-3z & 3-12z \end{bmatrix}$, for x, y, z.

3) Given the matrices $C=\begin{bmatrix} 1 & 4 & 2 \\ 4 & 1 & 5 \end{bmatrix}$, $D=\begin{bmatrix} 1 & 4 & 2 \\ 1 & 0 & -1 \\ 4 & 2 & 10 \end{bmatrix}$, $E=\begin{bmatrix} 4 & 1 & 1 \\ -1 & 0 & 1 \\ 4 & 1 & -1 \end{bmatrix}$.

Evaluate:

a. $E+D$

b. $E-D+I_3$

c. $5C$

d. $2D+4E\cdot I_3$

e. $2tr\left(D^2-2E\right)$

Given the matrices $A=\begin{bmatrix} 4 & 0 \\ 1 & 2 \\ -1 & 1 \end{bmatrix}$, $B=\begin{bmatrix} 4 & 1 \\ 0 & -2 \end{bmatrix}$, $C=\begin{bmatrix} 1 & 4 & 2 \\ 4 & 1 & 5 \end{bmatrix}$, $D=\begin{bmatrix} 1 & 4 & 2 \\ 1 & 0 & -1 \\ 4 & 2 & 10 \end{bmatrix}$.

f. $4C^T+A$

g. $\frac{1}{2}A^T+\frac{1}{4}C$

h. $I_2\cdot B\cdot C$

i. trC^TC

j. $D\cdot A\cdot B\cdot C$

4) Rewrite each of the following SLEs in matrix form $A\underline{x}=\underline{b}$:

a. $\begin{cases} 2x+y-z=3 \\ x+2y-4z=5 \\ 6x+4y+z=2 \end{cases}$

b. $\begin{cases} 2x-3y+z+t=1 \\ 4x+y+2z=4 \\ y+z+t=1 \\ x-4z-2y=10 \end{cases}$

5) Given: $A=\begin{bmatrix} 4 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & -6 & 3 \end{bmatrix}$ $\quad \underline{x}=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\quad \underline{b}=\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Express each of the following as an SLE:

a. $A\underline{x}=\underline{b}$ b. $A\underline{x}=4\underline{x}+\underline{b}$ c. $A\underline{x}=-k\underline{x}+\underline{b}$

d. $A\underline{x}=\underline{x}$ e. $A^T\underline{x}=2\underline{x}+3\underline{b}$

6) Recall that a square matrix A is *symmetric* if $A^T=A$ and *antisymmetric* if $A^T=-A$.

a. Given a square matrix A, which of the following must be true?

i. $A\cdot A^T$ is symmetric.

ii. $A+A^T$ is symmetric.

iii. $A-A^T$ is antisymmetric.

b. Given 2 antisymmetric $n\times n$ matrices A and B, which of the following must be true:

i. $BABABA$ is antisymmetric ii. A^2-B^2 is symmetric iii. A^2+B is symmetric

c. Given 2 antisymmetric $n\times n$ matrices A and B, which satisfy $AB=-BA$.
Which of the following must be true:

i. AB^3 is antisymmetric ii. AB^2 is symmetric iii. $(A-B)^2$ is symmetric

d. Given 2 antisymmetric $n\times n$ matrices A and B, which satisfy $AB=BA$. Prove:

i. AB is antisymmetric ii. $AB+B$ is antisymmetric

e. Given that the $n\times n$ matrices A, B, AB are symmetric, prove that $A^4B^4=B^4A^4$.

Answer Key:

1) a. $C_{4\times6}$ b. Undefined. c. $B_{4\times2}$ d-e. Undefined.

f. $C_{6\times6}$ g. $B_{6\times6}$ h. Undefined. i. $B_{6\times2}$ j. $C_{6\times6}$

2) $x=2,\ y=1,\ z=-1$

3) a. $\begin{bmatrix}5&5&3\\0&0&0\\8&3&9\end{bmatrix}_{3\times3}$ b. $\begin{bmatrix}4&-3&-1\\-2&1&2\\0&-1&-10\end{bmatrix}_{3\times3}$ c. $\begin{bmatrix}5&20&10\\20&5&25\end{bmatrix}$

d. $\begin{bmatrix}18&12&8\\-2&0&2\\24&8&16\end{bmatrix}$ e. 230 f. $\begin{bmatrix}8&16\\17&6\\7&21\end{bmatrix}_{3\times2}$ g. $\begin{bmatrix}2\frac{1}{4}&1\frac{1}{2}&0\\1&1\frac{1}{4}&1\frac{3}{4}\end{bmatrix}$

h. $\begin{bmatrix}8&17&13\\-8&-2&-10\end{bmatrix}_{2\times3}$ i. 63 j. $\begin{bmatrix}-32&82&-22\\48&87&75\\-48&108&-36\end{bmatrix}$

4) a. $\underbrace{\begin{bmatrix}2&1&-1\\1&2&-4\\6&4&1\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x\\y\\z\end{bmatrix}}_{\underset{\sim}{x}}=\underbrace{\begin{bmatrix}3\\5\\2\end{bmatrix}}_{\underset{\sim}{b}}$ b. $\underbrace{\begin{bmatrix}2&-3&1&1\\4&1&2&0\\0&1&1&1\\1&-2&-4&0\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x\\y\\z\\t\end{bmatrix}}_{\underset{\sim}{x}}=\underbrace{\begin{bmatrix}1\\4\\1\\10\end{bmatrix}}_{\underset{\sim}{b}}$

5) a. $\begin{cases}4x-2y+4z=1\\x-y+z=2\\x-6y+3z=3\end{cases}$ b. $\begin{cases}-2y+4z=1\\x-5y+z=2\\x-6y-z=3\end{cases}$ c. $\begin{cases}(4+k)x-2y+4z=1\\x+(-1+k)y+z=2\\x-6y+(3+k)z=3\end{cases}$

d. $\begin{cases}3x-2y+4z=0\\x-2y+z=0\\x-6y+2z=0\end{cases}$ e. $\begin{cases}2x+y+z=3\\-2x-3y-6z=6\\4x+y+z=9\end{cases}$

6) a. All three are true. b. (ii) Is true. c. All three are true.

d-e. Proved as shown in the videos.

Matrix Inverse and its Applications

Questions:

Find the inverse of the matrix:

1) $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$

2) $A=\begin{bmatrix}5 & 2\\ 7 & 3\end{bmatrix}$

3) $A=\begin{bmatrix}4 & 1.5\\ 2 & 1\end{bmatrix}$

4) $A=\begin{bmatrix}1 & 0 & 2\\ 4 & -1 & 8\\ 2 & 1 & 3\end{bmatrix}$

5) $A=\begin{bmatrix}2 & 1 & 1\\ 0 & 2 & -1\\ 5 & 2 & 3\end{bmatrix}$

6) $A=\begin{bmatrix}2 & -1 & 1\\ 3 & -2 & 2\\ 5 & -3 & 4\end{bmatrix}$

7) $A=\begin{bmatrix}1 & 0 & 0 & 0\\ 1 & 2 & 0 & 0\\ 1 & 2 & 3 & 0\\ 1 & 2 & 3 & 4\end{bmatrix}$

Find the values of k for which the following matrixes are invertible:

8) $\begin{bmatrix}1 & -1 & 1\\ 5 & -7 & k^2+3\\ 3 & -1 & k+3\end{bmatrix}$

9) $\begin{bmatrix}1 & 1 & 1 & 1 & k\\ 1 & 1 & 1 & k & 1\\ 1 & 1 & k & 1 & 1\\ 1 & k & 1 & 1 & 1\\ k & 1 & 1 & 1 & 1\end{bmatrix}$

10) Solve the following SLE by using the Inverse Matrix method: $\begin{cases}2x-y+z=3\\ 3x-2y+2z=5\\ 5x-3y+4z=11\end{cases}$.

11) Solve the following SLE by using the Inverse Matrix method: $\begin{matrix}x+4y+2z+4t=1\\ x+2y-z=0\\ y+z+t=1\\ x+3y-z-2t=0\end{matrix}$.

Answer Key:

1) $\begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$

2) $\begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$

3) $\begin{bmatrix} 1 & -1.5 \\ -2 & 4 \end{bmatrix}$

4) $\begin{bmatrix} -11 & 2 & 2 \\ 4 & -1 & 0 \\ 6 & -1 & -1 \end{bmatrix}$

5) $\begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ -10 & 1 & 4 \end{bmatrix}$

6) $\begin{bmatrix} 2 & -1 & 0 \\ 2 & -3 & 1 \\ -1 & -1 & 1 \end{bmatrix}$

7) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{4} & \frac{1}{4} \end{bmatrix}$

8) $k \neq -2$, $k \neq 1$

9) $k = 1$, $k = -4$

10) $x = 1,\ y = 2,\ z = 3$

11) $x = -13,\ y42,\ z = -5,\ t = 2$

Properties of the Matrix Inverse

Questions:

1) Assume that all the matrices are invertible and of order n, and extract X:

a. $AXC = D$ b. $A^{-1}XC = A^{-1}DC$ c. $P^{-1}X^{T}P = A$

d. $C^{-1}(A+X)D^{-2} = I$ e. $(A-AX)^{-1} = X^{-1}C$ f. $ABC^{T}X^{-1}BA^{T}C = AB^{T}$

2) Given that $B = \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}$ and that $B^{2}X(2B)^{-1} = B + I$. Find X.

All matrices are 2×2.

3) Given that $B^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 4 & -1 & 8 \\ 2 & 1 & 3 \end{bmatrix}$ and that $BYB^{T} = B^{-1} + B$. Find Y.

All matrices are 3×3.

4) Given that $A^{-1} = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$ and that $5A^{T}\,\mathrm{B}(\mathrm{I}+2\,\mathrm{A})^{-2} = (7A)^{-2}$. Find B.

All matrices are 2×2.

5) Prove the following:

a. Given: A is a square matrix satisfying $A^{2} - 5A - 2I = 0$.
Prove that A is invertible and express A^{-1} in terms of A and I.

b. Given: A is a square matrix satisfying $(A-3I)(A+2I) = 0$.
Prove that A is invertible and express A^{-1} in terms of A and I.

6) Prove the following:

a. Given: A is a square matrix satisfying $A^{2} - 5A - 2I = 0$.
Prove that A is invertible and express A^{-1} in terms of A and I.

b. Given: A is a square matrix satisfying $(A-3I)(A+2I) = 0$.
Prove that A is invertible and express A^{-1} in terms of A and I.

7) Given: $A=\begin{bmatrix}-1 & 3 & 0\\ 3 & -1 & 0\\ -2 & -2 & 6\end{bmatrix}$, $p(x)=x^3-4x^2-20x+48$.

a. Compute $p(A)$.

b. Using part a. above, prove that A is invertible and express A^{-1} in terms of A and I.

8) Given: A is a square matrix satisfying $A^4=0$.

a. Prove that A is noninvertible.

b. Prove that the matrix $I-A$ is invertible and find its inverse.

9) Given: $\begin{cases}P^{-1}AP=B\\ Q^{-1}BQ=C\end{cases}$.

Prove that there exists an invertible matrix D such that $D^{-1}AD=C$.

Answer Key:

1) 1. $X=A^{-1}DC^{-1}$ 2. $X=D$ 3. $X=\left(P^{-1}\right)^T A^T P^T$

4. $X=CD^2-A$ 5. $X=\left(A+C^{-1}\right)^{-1}A$ 6. $X=BA^TC\cdot\left(B^T\right)^{-1}BC^T$

2) $X=\begin{bmatrix}20 & -4\\ -8 & 4\end{bmatrix}$

3) $Y=\begin{bmatrix}22 & 86 & 38\\ 64 & 246 & 114\\ 56 & 222 & 92\end{bmatrix}$

4) $B=\frac{1}{245}\begin{bmatrix}264 & 450\\ 448 & 768\end{bmatrix}$

5) Proved as shown in the video.

6) Proved as shown in the video.

7) a. $P(A)=0$ b. Proved as shown in the video

8) Proved as shown in the video.

9) Proved as shown in the video.