proprep™

Workbook

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Moment Generating Function

Moment Generating Function

Questions:

1) The following probability function is given for a discrete random variable:

x	1	2	3
$p(x)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

a. Find the moment generating function.

b. Derive the expectation from the moment generating function.

2) Find the moment generating function of the binomial distribution $X \sim \mathrm{B}(n, p)$, and find the first and second moments of the function.

3) Find the moment generating function of the geometric distribution $X \sim \mathrm{G}(p)$ and calculate the expectation of the probability distribution from the moment generating function.

4) Find the moment generating function of the Poisson distribution $X \sim \mathrm{P}(\lambda)$. Find the first and second moments of the probability distribution

5) Let X be a random variable with the following density function:
$f(X) = Ae^{-x}$ if $X > 0$ or $X = 0$ and $X < 7$ or $X = 7$, and 0 otherwise.

a. Find the value of A.

b. Find the moment generating function of X.

6) Let X be a random variable with an expectation of 5 and a variance of 16.
Let $\mathrm{M}_x(t)$ be the moment generating function of X.
Y is a random variable with a moment generating function $\mathrm{M}_y(t)$.
Assume $\mathrm{M}_y(t) = t \cdot \mathrm{M}_x(t)$
Calculate the expectation and variance of Y.

Answer Key:

1) a. $\frac{1}{2}e^{t}+\frac{1}{3}e^{2t}+\frac{1}{6}e^{3t}$ b. $\mathrm{E}(x)=1\frac{2}{3}$

2) $\mathrm{M}_x(t):=\left(e^{t}\cdot p+1-p\right)^{n}$;

$\mathrm{E}(x)=\mathrm{M}_x'(t)=n\cdot p$; $\mathrm{E}\left(x^2\right)=\mathrm{M}_x''(t)=np\left[(n-1)p+1\right]$

3) $\mathrm{M}_x(t)=\frac{e^{t}p}{1-e^{t}\cdot(1-p)}$; $\mathrm{E}(x)=M_x'(t)=\frac{1}{p}$

4) $\mathrm{M}_x(t)=e^{\lambda\left(e^{t}-1\right)}$; $\mathrm{E}(x)=M_x'(t)=\lambda$; $\mathrm{E}\left(x^2\right)=\mathrm{M}_x''(t)=\lambda(\lambda+1)$

5) a. $A=\frac{1}{1-e^{-7}}$ b. $\mathrm{M}_x(t)=\left(\frac{e^{7}}{e^{7}-1}\right)\frac{\left(e^{7(t-1)}\right)}{(t-1)}$

6) $\mathrm{E}(Y)=1$, $\mathrm{V}(Y)=9$

Appendix

Probability Distribution	Density Function $f_x(t)$ $f_X(t)$	Cumulative Distribution Function $F_x(t)$	$E(x)$	$Var(x)$	$M_X(t)$
Uniform $U(a,b)$	$f_x(t)=\frac{1}{b-1}$ if $a<t$ or $a=t$ and $t<b$ or $t=b$ 0 otherwise	$F_x(t)=\begin{cases}0 & t<a \\ \frac{t-a}{b-a} & a\le t\le b \\ 1 & t<b\end{cases}$	$\frac{b-a}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{tb}-e^{ta}}{t(b-a)}$
Exponential $\exp(\lambda)$	$f_x(t)=\lambda e^{-\lambda t}$ if $t>0$ or $t=0$ 0 otherwise	$F_x(t)=1-e^{-\lambda t}$ if $t>0$ or $t=0$ 0 otherwise	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda-t}$
Normal $N(\mu,\sigma^2)$	$f_X(t)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(t-\mu)^2}{2\sigma^2}}$	$\phi\left(\frac{t-\mu}{\sigma}\right)$	μ	σ^2	$e^{\mu t+\frac{\sigma^2t^2}{2}}$

Random Variable	Significance	$P_X(x)$	$E(x)$	$Var(x)$	$M_X(t)$
Binomial $Bin(n,p)$	The same Bernoulli trial is independently repeated n times. p is the probability of success, $1-p=q$ is the probability of failure x is the number of successes	$\binom{n}{x}p^xq^{n-x}$ $x=0,1,...,n$	$n\cdot p$	$n\cdot p\cdot q$	$\left[pe^t+q\right]^n$
Geometric $G(p)$	The same Bernoulli trial is independently repeated until the first success. x is the number of trials until the first success.	pq^{x-1} $x=1,2,...,\infty$	$\frac{1}{p}$	$\frac{q}{p^2}$	$\frac{pe^t}{1-qe^t}$
Poisson $Pois(\lambda)$	x is the number of appearances in a unit of time. The length of time receives values $0,1,...,\infty$	$e^{-\lambda}\frac{\lambda^x}{x!}$	λ	λ	$e^{\lambda(e^t-1)}$

Characteristics of the Moment Generating Function

Questions:

1) Assume that $X_i \sim P(\lambda)$ are independent.

a. Find the moment generating function of $\sum_{i=1}^{n} X_i$.

b. Prove that $\sum_{i=1}^{n} X_i \sim P(n \cdot \lambda)$.

2) Assume that $X \sim P(\lambda = 10)$, $Y \sim P(\lambda = 2)$ and X and Y are independent.

a. Find the moment generating function of T .

b. Prove that $T \sim P(\lambda = 12)$.

c. Prove that $X \mid T = 8 \sim B\left(n = 8, p = \frac{5}{6}\right)$, i.e. that the probability distribution of X, given $T = 8$, is binomial with parameters $n = 8$ and $p = \frac{5}{6}$.

3) Assume that $X_i \sim \exp(1)$ $i = 1, 2, ..., n$ and the variables are independent.

We define $T = \sum_{i=1}^{n} X_i$.

a. Find the moment generating function of T.

b. Calculate the expectation and variance of T.

c. Let $Z = \frac{T - E(T)}{\sigma(T)}$, i.e. Z is the standardization of T.

Find the moment generating function of Z .

4) The moment generating function of the normal probability distribution is given by the following formula: $M_X(T) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$ for any T Where $X \sim N(\mu, \sigma^2)$

a. Prove that if $Y = 2X$, then $Y \sim N(2\mu, 4\sigma^2)$.

b. Prove that if $T = X_1 + X_2$ and X_1 and X_2 are independent variables with the same normal probability distribution, then $T \sim N(2\mu, 2\sigma^2)$.

Answer Key:

1) a. $M_T(t)=e^{n\lambda(e^t-1)}$

b. since a. has the same format as $M_x(t)=e^{\lambda(e^t-1)}$, we know that

$$\sum_{i=1}^{n} x_i = T \sim P(n\lambda)$$

2) a. $M_T(t)=e^{12(e^t-1)}$ b. $T=X+Y \sim P(\lambda=12)$

c. $P(x=k|T=8)=\frac{P(x=k\cap T=8)}{P(T=8)}$, we know that $T=X+Y=8$ so $Y=8-X$

and we can write it as $=\frac{P(X=k\cap Y=8-k)}{P(T=8)}=\frac{P(X=k)\cdot P(Y=8-k)}{P(T=8)}$

$$\frac{\frac{e^{-10}\cdot 10^k}{k!}\cdot\frac{e^{-2}\cdot 2^{8-k}}{(8-k)!}}{\frac{e^{-12}\cdot 12^8}{8!}}=\frac{e^{-10}\cdot 10^k}{k!}\cdot\frac{e^{-2}\cdot 2^{8-k}}{(8-k)!}\cdot\frac{8!}{e^{-12}\cdot 12^8}=\frac{8!}{k!(8-k)!}\cdot\frac{10^k\cdot 2^8}{2^k\cdot 12^8}$$

$$\binom{8}{k}5^k\cdot\left(\frac{1}{6}\right)^8=\binom{8}{k}\frac{5^k}{6^k}\left(\frac{1}{6}\right)^8\cdot 6^k=\binom{8}{k}\left(\frac{5}{6}\right)^k\left(\frac{1}{6}\right)^8\cdot\frac{1}{6}^{-k}$$

$$=\binom{8}{k}\left(\frac{5}{6}\right)^k\left(\frac{1}{6}\right)^{k-8} \sim B\left(n=8, p=\frac{5}{6}\right)$$

3) a. $M_T(t)=\left(\frac{1}{1-t}\right)^n$ b. $E(T)=n$ $E(T^2)=n(n+1)$ $V(T)=n$

c. $M_Z(t)=e^{-n^{\frac{1}{2}}\cdot t}\left(\frac{1}{1-n^{-\frac{1}{2}}\cdot t}\right)^n$

4) a. Y is a linear transformation, so $Y=aX+b$, and therefore $a=2$, b=0.

Since $M_{aX+b}(t)=e^{bt}\cdot M_X(at)$, we get $M_y(t)=e^{0\cdot t}\cdot e^{\mu\cdot 2t+\frac{\sigma^2(2t)^2}{2}}=1\cdot e^{2\mu t+\frac{4\sigma^2t^2}{2}}=e^{(2\mu)t+\frac{(4\sigma^2)t^2}{2}}$

which means $y \sim N(2\mu, 4\sigma^2)$.

b. We know that $T=X_1+X_2$, and also that $X_1 \sim N(\mu,\sigma^2)$ and the same for X_2.

Using the formula of moment generating function of two random variables:

$M_{T=X_1+X_2}(t)=\left(e^{\mu t+\frac{\sigma^2t^2}{2}}\right)^2=e^{(2\mu)t+\frac{(2\sigma^2)t^2}{2}}$ which is the same format of the given normal probability distribution, so $T \sim N(2\mu, 2\sigma^2)$.