proprep™

Workbook

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Phase Planes

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Questions

1) For the matrix: $A=\begin{bmatrix}-4 & 1\\ 3 & -2\end{bmatrix}$, consider the ODE system: $x'=Ax$, where $x(t)=\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$ denotes the position of a particle in the plane at time t.

a. Find the general solution $x(t)$ of this differential equation.

b. Classify the critical point and sketch a phase portrait of the system in the plane.

2) For the matrix: $A=\begin{bmatrix}1 & 2\\ 3 & 0\end{bmatrix}$, consider the ODE system: $x'=Ax$, where $x(t)=\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$ denotes the position of a particle in the plane at time t.

a. Find the general solution $x(t)$ of this differential equation.

b. Classify the critical point and sketch a phase portrait of the system in the plane.

3) For the matrix: $A=\begin{bmatrix}4 & 3\\ 8 & 2\end{bmatrix}$, consider the ODE system: $x'=Ax$, where $x(t)=\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$ denotes the position of a particle in the plane at time t.

a. Find the general solution $x(t)$ of this differential equation.

b. Classify the critical point and sketch a phase portrait of the system in the plane.

4) For the nonsingular matrix: $A=\begin{bmatrix}1 & -2\\ 2 & 1\end{bmatrix}$,

consider the ODE system: $x'=Ax$, where $x(t)=\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$.

a. Find the general solution $x(t)$ of the system.

b. Classify the critical point (at the origin) and sketch a portrait of the system in the phase plane.

5) Consider the nonlinear planar system of ODEs: $\begin{cases} \dfrac{dx}{dt} = x(4-x-3y) \\ \dfrac{dy}{dt} = y(3-x-2y) \end{cases}$.

a. Find the critical points (AKA equilibrium solutions).
b. Near each critical point, find the corresponding linear system and determine the type of its critical point (of the linear system).
c. Which critical points of the <u>original</u> system are stable? Unstable? Asymptotically stable?
d. Try to give a rough sketch of the phase portrait for the system.
e. For $x(t) \geq 0$, $y(t) \geq 0$, suppose the system models the populations $x(t)$, $y(t)$ of two species.
Does the model suggest the species could coexist in a stable way?

6) Let A be a 2×2 real matrix with eigenvalues $\lambda_1 = -1,\ \lambda_2 = -5$ and corresponding eigenvectors $v_{\lambda_1=-1} = \begin{bmatrix} 1 \\ 3 \end{bmatrix},\ v_{\lambda_2=-5} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

a. Give a general solution to the linear planar system: $x' = Ax$, where $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$.
b. Find an invertible matrix T and a diagonal matrix D, such that $T^{-1}AT = D$.
c. Compute e^{tA}.
d. Sketch a phase portrait (with arrows) and classify the type and stability of the origin. [Recall that the origin is the only critical point of such a system]

7) Consider the nonlinear planar system of autonomous ODEs $\begin{cases} x' = 2xy = F(x,y) \\ y' = 2x - y^2 + 9 = G(x,y) \end{cases}$.

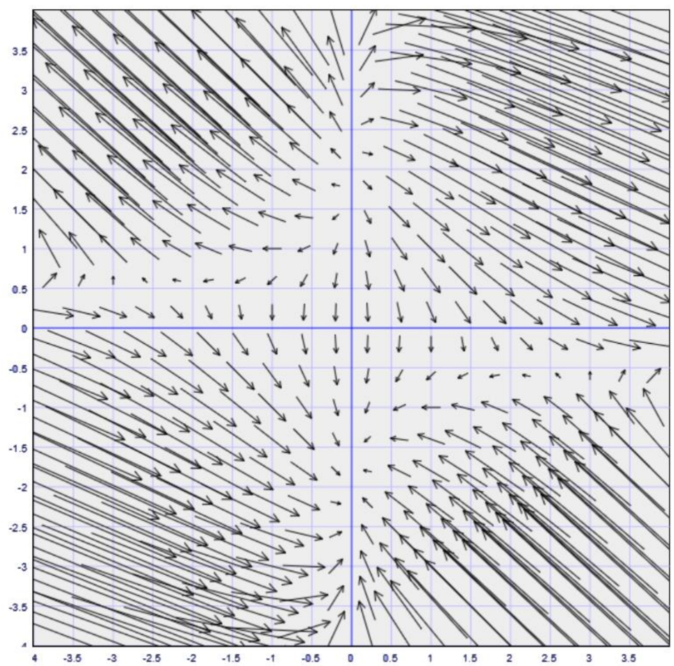
a. Find a nonlinear function $H(x,y)$, such that every orbit (trajectory) of the system satisfy $H(x,y) = c$ for some constant c.
b. Find the critical points.
Which of these points are saddle points?
c. For each value of c that corresponds to a saddle point, sketch the level set $H(x,y) = c$, plus the critical points.

8) Consider the nonlinear planar system of autonomous ODEs $\begin{cases} x' = -3x + y = F(x,y) \\ y' = -5x - y + 4x^2 = G(x,y) \end{cases}$.

a. Find the stationary (critical) points of the system.
b. Find the Jacobian matrix at each stationary point.
c. Classify the type and stability of each stationary point.
d. Sketch a phase portrait of the system, that shows its behavior near each stationary point (Use arrows to mark the sketched orbits).

9) Consider the nonlinear planar system of ODEs: $\begin{cases} \dfrac{dx}{dt} = \underbrace{5xy + x^2}_{F(x,y)} \\ \dfrac{dy}{dt} = \underbrace{y^2 - 3xy - 4}_{G(x,y)} \end{cases}$

and the direction field plot (by MATLAB), as shown:



a. What are the stationary (critical) points of the system?
b. Classify the type and stability of each stationary point.
c. Predict the behavior, as $t \to \infty$, of the solution passing through $(2.5, 0)$. Sketch your predicted solution in the phase plane.

10) Consider the nonlinear autonomous system $\begin{cases} x' = x^2 - 9 = F(x,y) \\ y' = 2xy \quad = G(x,y) \end{cases}$.

a. Find the critical points of the system.
b. Write down the linearization of the system at each critical point.
c. Classify the type and stability of each critical point.
d. Sketch a phase portrait of the system (with arrows on the sketched orbits).

11) Consider the ODE system: $x' = \begin{bmatrix} 9 & 2 \\ -8 & 1 \end{bmatrix} x$, where $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$.

Classify the critical point and sketch a phase portrait of the system in the plane.

Answer Key

1) a. $x(t)=c_1e^{-t}\begin{bmatrix}1\\3\end{bmatrix}+c_2e^{-5t}\begin{bmatrix}-1\\1\end{bmatrix}$ b. $\begin{bmatrix}0\\0\end{bmatrix}$ node point which is asymptotically stable.

2) a. $x(t)=c_1e^{-2t}\begin{bmatrix}-2\\3\end{bmatrix}+c_2e^{3t}\begin{bmatrix}3\\1\end{bmatrix}$ b. $\begin{bmatrix}0\\0\end{bmatrix}$ saddle point which is unstable.

3) a. $x(t)=c_1e^{-2t}\begin{bmatrix}-1\\2\end{bmatrix}+c_2e^{8t}\begin{bmatrix}3\\4\end{bmatrix}$ b. $\begin{bmatrix}0\\0\end{bmatrix}$ saddle point which is unstable.

4) a. $x(t)=c_1e^{t}\begin{bmatrix}\cos 2t\\-\sin 2t\end{bmatrix}+c_2e^{t}\begin{bmatrix}\sin 2t\\\cos 2t\end{bmatrix}$ b. $\begin{bmatrix}0\\0\end{bmatrix}$ spiral point which is unstable.

5) a. $(0,0)$, $\left(0,\frac{3}{2}\right)$, $(4,0)$, $(1,1)$

b. We find: $A(x,y)=\begin{bmatrix}4-2x-3y & -3x\\-y & 3-x-4y\end{bmatrix}$. Then:

$A(0,0)=\begin{bmatrix}4 & 0\\0 & 3\end{bmatrix}$ unstable node. $A\left(0,\frac{3}{2}\right)=\begin{bmatrix}-\frac{1}{2} & 0\\-\frac{3}{2} & -3\end{bmatrix}$, asymptotically stable node.

$A(4,0)=\begin{bmatrix}-4 & -12\\0 & -1\end{bmatrix}$, asymptotically stable node. $A(1,1)=\begin{bmatrix}-1 & -3\\-1 & -2\end{bmatrix}$, saddle (unstable).

c. $(0,0)$, $(1,1)$ are unstable, $\left(0,\frac{3}{2}\right)$, $(4,0)$ are asymptotically stable.

d. Solution in the recording. e. No.

6) a. $x(t)=c_1e^{-t}\begin{bmatrix}1\\3\end{bmatrix}+c_2e^{-5t}\begin{bmatrix}-1\\1\end{bmatrix}$ b. $T=\begin{bmatrix}1 & -1\\3 & 1\end{bmatrix}$, $D=\begin{bmatrix}-1 & 0\\0 & -5\end{bmatrix}$

c. $e^{tA}=\frac{1}{4}\begin{bmatrix}e^{-t}+3e^{-5t} & e^{-t}-e^{-5t}\\3e^{-t}-3e^{-5t} & 3e^{-t}+e^{-5t}\end{bmatrix}$ d. Solution in the recording.

7) a. $H(x,y)=x^2-y^2x+9x$ b. $(4.5,0)$ Circle point. $(0,\pm 3)$ Saddle points.

c. Solution in the recording.

8) a. $(0,0)$, $(2,6)$ b. $A(0,0)=\begin{bmatrix}-3 & 1\\ -5 & -1\end{bmatrix}$, $A(2,6)=\begin{bmatrix}-3 & 1\\ 11 & -1\end{bmatrix}$

c. $(0,0)$ spiral point, asymptotically stable. $(2,6)$ saddle point, unstable.

d. Solution in the recording.

9) a. $(0,2)$, $(0,-2)$, $(-2.5,0.5)$, $(2.5,-0.5)$

b. $(0,2)$ node, unstable. $(0,-2)$ node, asymptotically stable.

$(-2.5,0.5)$ saddle, unstable. $(2.5,-0.5)$ saddle, unstable.

c. Solution in the recording.

10) a. $(3,0)$, $(-3,0)$

b. Linearization at $(3,0)$: $x'=\begin{bmatrix}6 & 0\\ 0 & 6\end{bmatrix}x$ Linearization at $(-3,0)$: $x'=\begin{bmatrix}-6 & 0\\ 0 & -6\end{bmatrix}x$.

c. $(3,0)$ proper node, unstable, $(-3,0)$ proper node, asymptotically stable.

d. Solution in the recording.

11) Solution in the recording.