

# Workbook



## Table of Contents

Second Order Linear Equations .....	2
Missing $x$ or $y$ , Reduction of Order .....	2
Linear, Homogeneous, Constant Coefficients .....	3
Linear, Nonhomogeneous, Constant Coefficients - Method of Undetermined Coefficients ..	4
Linear, Nonhomogeneous, Constant Coefficients - Method of Variation of Parameters .....	5
Linear, Homogeneous, Non-Constant Coefficients - 2nd Solution Method .....	6
The Wronskian and Its Uses.....	7
Sturm-Liouville Problems.....	9



# Second Order Linear Equations

## Missing x or y, Reduction of Order

---

### Questions

Solve the following equations:

1)  $x^2 y'' + xy' = \frac{1}{x} \quad (x \neq 0)$

2)  $y'' \tan x - 1 = y', \quad (\cos x \neq 0)$

3)  $2xy' y'' - (y')^2 + 1 = 0$

4)  $y'' x \ln x = y'$

5)  $xy'' = x^2 e^x + y'$

6)  $y \cdot y'' + (y')^2 = 0$

7)  $2y'' y - (y')^2 = 1$

8)  $y'' \tan y = 2(y')^2 \quad (\cos y \neq 0)$

### Answer Key

1)  $y = \frac{1}{x} + C_1 \cdot \ln x + C_2$

2)  $y = -x + C_1 \cdot \cos x + C_2$

3)  $y = \pm x + C_3$

4)  $y = C_3$

5)  $y = e^x (x-1) + C_1 \frac{x^2}{2} + C_2$

6)  $\frac{y^2}{2} = cx + k, \quad y = c$

7)  $y = \frac{1}{c} \left[ \frac{c^2 (x+k)^2}{4} + 1 \right]$

8)  $\cot y = -(cx + k), \quad y = c$

### Linear, Homogeneous, Constant Coefficients

---

#### Questions

Solve the following equations:

1)  $y'' - 100y = 0$

2)  $y'' - 4y' = 0$

3)  $y'' - 8y' + 7y = 0$

4)  $4z'' + z' - 5z = 0$  ;  $z(0) = 1$  ,  $z'(0) = 1$

5)  $y'' - 2y' + y = 0$

6)  $4\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + x = 0$

7)  $y'' + 10y' + 125y = 0$

8)  $y'' + 4y = 0$

9)  $y'' - 2y' + 10y = 0$  ;  $y(0) = 0$  ,  $y'(0) = 3$

#### Answer Key

1)  $y = c_1e^{10x} + c_2e^{-10x}$

2)  $y = c_1 + c_2e^{4x}$

3)  $y = c_1e^x + c_2e^{7x}$

4)  $z = e^x$

5)  $y = c_1e^x + c_2xe^x$

6)  $x = c_1e^{-0.5t} + c_2te^{-0.5t}$

7)  $y = e^{-5x} [c_1 \cos 10x + c_2 \sin 10x]$

8)  $y = c_1 \cos 2x + c_2 \sin 2x$

9)  $y = e^x \sin 3x$

## Linear, Nonhomogeneous, Constant Coefficients - Method of Undetermined Coefficients

---

### Questions

Solve the following equations:

- |  |  |
|--|--|
| 1) $y'' + 5y' + 6y = 22x + 6x^2$                   | 2) $y'' - 2y' + y = e^{2x}; y(0) = 2, y'(0) = 7$ |
| 3) $y'' - y' - 2y = 4\sin 2x$                      | 4) $y'' - 2y = xe^{-x}$                          |
| 5) $y'' - y = 3e^{2x} \cos x$                      | 6) $z'' + z = \sin x$                            |
| 7) $y'' - 3y' + 2y = 2x^2 + e^x + 2xe^x + 4e^{3x}$ | 8) $y'' + 3y' = 9x$                              |
| 9) $y'' - 3y' + 2y = e^x$                          | 10) $y'' - 2y' = 6x^2 - 2x$                      |
| 11) $x'' + 5x' + 6x = e^{-t} + e^{-2t}$            | 12) $y'' + 2y' + 5y = e^{-x} \sin 2x$            |

### Answer Key

- |  |   |
|--|---|
| 1) $y = c_1 e^{-3x} + c_2 e^{-2x} + x^2 + 2x - 2$                                      | 2) $y = e^x + 4xe^x + e^{2x}$                                 |
| 3) $y = c_1 e^{2x} + c_2 e^{-x} + \frac{1}{5} \sin 2x - \frac{3}{5} \cos 2x$           | 4) $y = c_1 e^{-\sqrt{2}x} + c_2 e^{\sqrt{2}x} + (2-x)e^{-x}$ |
| 5) $y = c_1 e^{-x} + c_2 e^x + \frac{3}{10} e^{2x} \cos x + \frac{3}{5} e^{2x} \sin x$ | 6) $z = c_1 \cos x + c_2 \sin x - \frac{1}{2} x \cos x$       |
| 7) $y = c_1 e^x + c_2 e^{2x} + x^2 + 3x + 3.5 - x^2 e^x - 3xe^x + 2e^{3x}$             | 8) $y = c_1 + c_2 e^{-3x} + \frac{3}{2} x^2 - x$              |
| 9) $y = c_1 e^x + c_2 e^{2x} - xe^x$   | 10) $y = c_1 e^{-3x} + c_2 e^{-2x} - x^2 - x - x^3$           |
| 11) $x = c_1 e^{-2t} + c_2 e^{-3t} + \frac{1}{2} \cdot e^{-t} + te^{-2t}$              |   |
| 12) $y = e^{-x} [c_1 \cos 2x + c_2 \sin 2x] - \frac{1}{4} x \cdot e^{-x} \cos 2x$      |   |

## Linear, Nonhomogeneous, Constant Coefficients - Method of Variation of Parameters

---

### Questions

Solve the following equations:

1)  $y'' + y = \frac{1}{\sin x}$

2)  $y'' + 4y' + 4y = e^{-2x} \ln x$

3)  $y'' + 2y' + y = 3e^{-x} \sqrt{x+1}$

4)  $y'' - 2y' + y = \frac{e^x}{x}$  ;  $y(1) = 0$  ,  $y'(1) = 0$

5)  $y'' - 3y' + 2y = \frac{1}{1+e^{-x}}$

6)  $y'' + 4y = \sec 2x$

### Answer Key

1)  $y = c_1 \cos x + c_2 \sin x - \cos x \cdot x + \sin x \cdot \ln |\sin x|$

2)  $y = c_1 e^{-2x} + c_2 x e^{-2x} - e^{-2x} \frac{x^2}{2} \left[ \ln x - \frac{1}{2} \right] + x^2 e^{-2x} [\ln x - 1]$

3)  $y = c_1 e^{-x} + c_2 x e^{-x} - e^{-x} \left[ \frac{6(\sqrt{x+1})^5}{5} - \frac{6(\sqrt{x+1})^3}{3} \right] + x e^{-x} [2(x+1)^{3/2}]$

4)  $y = e^x - x e^x + x e^x \ln x$  ,  $(x > 0)$

5)  $y = c_1 e^x + c_2 e^{2x} + e^x \ln(1+e^{-x}) + e^{2x} [\ln(1+e^{-x}) - (1+e^{-x})]$

6)  $y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{4} \cos 2x \ln |\cos 2x| + \frac{1}{2} \sin 2x \cdot x$

## Linear, Homogeneous, Non-Constant Coefficients - 2nd Solution Method

---

### Questions

Solve the following equations:

1)  $y'' + \tan x \cdot y' - (2 \tan x + 4)y = 0$

2)  $(1 - x^2)y'' + 2xy' - 2y = 0, (x > 0)$

3)  $(1 + x^2)y'' - 1.5xy' + y = 0, (x > 2)$

### Answer Key

1)  $y = c_1 e^{2x} + c_2 e^{-2x} (\sin x - 4 \cos x)$

2)  $y = c_1 x + c_2 (x^2 + 1)$

3)  $y = c_1 (x^2 - 2) + c_2 (x^2 - 2)h(x)$  where  $h(x)$  is defined in the recording.

### The Wronskian and Its Uses

---

#### Questions

Solve the following equations:

- 1) Is it possible that  $y_1(x) = e^x$ ,  $y_2(x) = \sin x$  are two solutions of a linear homogeneous 2<sup>nd</sup> order ODE  $y'' + p(x)y' + q(x)y = 0$ , with continuous coefficients  $p(x), q(x)$  on the interval  $[0, \pi]$ ?
  
- 2) Answer the following questions:
  - a. Show that the functions  $y_1(x) = \sin x^2$ ,  $y_2(x) = \cos x^2$  are l.i. solutions of the equation  $xy'' - y' + 4x^3y = 0$  on any interval.
  - b. Show that the Wronskian is zero only for  $x = 0$ .
  - c. Joe claims that we have a contradiction to one of our propositions.
    - i. Which proposition is Joe referring to?
    - ii. Is Joe right?
  
- 3) It is easily verified that the functions  $y_1(x) = xe^x$ ,  $y_2(x) = e^{-x}$ , are solutions of the equation  $y'' - \frac{2}{1+2x}y' - \frac{2x+3}{1+2x}y = 0$  on the interval  $(-\frac{1}{2}, \infty)$ . Are these functions l.i. on the interval?
  
- 4) We're given two functions  $y_1 = x^3$ ,  $y_2 = |x^3|$ , on the interval  $I = [-4, 4]$ .
  - a. Compute the Wronskian of the functions on  $I$ .
  - b. Are the functions l.i. on  $I$ ?
  - c. Could the functions be two solutions of an ODE  $y'' + p(x)y' + q(x)y = 0$ , with continuous coefficients?
  - d. Note that our functions are solutions of the ODE  $xy'' - 2y' = 0$  on  $I$ . Does that contradict the result of part c?



5) Answer the following questions:

- a. Let  $y_1(x)$ ,  $y_2(x)$  be functions, which are twice continuously differentiable on an interval  $I$  and such that their Wronskian is nonzero on  $I$ .

Prove that there exists an ODE  $y'' + p(x)y' + q(x)y = 0$  with continuous coefficients on  $I$ , such that  $y_1(x)$ ,  $y_2(x)$  are two of its solutions.

- b. Find an equation  $y'' + p(x)y' + q(x)y = 0$  with continuous coefficients on  $x > 0$ , such that  $y_1(x) = x^2$ ,  $y_2(x) = x^4$  are two of its solutions.

### Answer Key

1)  $\cos x = \sin x$

- 2) a.  $W(x) = -2x$       b. This was already shown at a.      c. Solution in the recording.

3) Solution in the recording.

- 4) a.  $W = 0$  always.      b. Solution in the recording.      c. No      d. No contradiction.

- 5) a. Solution in the recording.      b.  $y'' - \frac{5}{x}y' + \frac{8}{x^2}y = 0$ , ( $x > 0$ )

### Sturm-Liouville Problems

---

#### Questions

Solve the following equations:

$$1) \begin{cases} y'' + \lambda y = 0, & 0 \leq x \leq 1 \\ y'(0) = 0 \\ y'(1) = 0 \end{cases}$$

$$2) \begin{cases} y'' + \lambda y = 0, & 0 \leq x \leq 1 \\ y(0) = 0 \\ y(1) + y'(1) = 0 \end{cases}$$

$$3) \begin{cases} y'' + \lambda y = 0, & 0 \leq x \leq 1 \\ y(0) + y'(0) = 0 \\ y(1) = 0 \end{cases}$$

$$4) \begin{cases} y'' + \lambda y = 0, & 0 \leq x \leq l \\ y(0) = 0 \\ y'(l) = 0 \end{cases}$$

$$5) \begin{cases} y'' + \lambda y = 0, & 0 \leq x \leq \pi \\ y'(0) = 0 \\ y(\pi) = 0 \end{cases}$$

$$6) \begin{cases} y'' - 2y' + (1 + \lambda)y = 0, & 0 \leq x \leq 1 \\ y(0) = 0 \\ y(1) = 0 \end{cases}$$

#### Answer Key

$$1) \quad y = 0$$

$$2) \quad y = 0$$

$$3) \quad y = 0$$

$$4) \quad y = 0$$

$$5) \quad y = 0$$

$$6) \quad y = 0$$