

# Workbook



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# Tangents, Normal Lines and Linear Approximation

## Tangent and Normal Lines – Basic Exercises

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### Questions

- 1) Find the equation of the line that is tangent to the curve  $f(x) = x^2 - 2x + 3$  at the point on the curve where  $x = 2$ .
- 2) Find the equation of the line that is normal to the curve  $f(x) = x^2 - 2x + 3$  at the point on the curve where  $x = 2$ .
- 3) Find the equation of the line that is tangent to the curve  $f(x) = x^3 - 4x + 2x - 5$  at the point on the curve where  $x = 1$ , Does the line intersect the curve at any other point?
- 4) Find the equation of the line that is tangent to the curve  $f(x) = x^3 - 6x + 2$  and are parallel to the line  $y = 6x - 2$ .
- 5) Find the equation of the line that is tangent to the curve  $f(x) = y = \sqrt{2x + 3}$  at the point on the curve where  $x = 3$ .
- 6) Find the equation of the line that is tangent to the curve  $f(x) = xe^{x^2}$  at the origin. Does the line intersect the curve at any other point?
- 7) Find the equation of the tangent to the curve  $f(x) = e^{\sin(4x)}$  at the point on the curve where  $x = \pi$ .
- 8) Find the equation of the lines that are normal to the curve  $f(x) = \frac{2x}{1-x}$  and parallel to the line  $y = -2x$ .

- 9) Find the equation of the line that is tangent to the curve  $f(x) = x \ln x$  at the point on the curve where  $x = e$ .
- 10) Find the equation of the line that is tangent to the curve  $f(x) = e^{x-1} \ln x$  at the point on the curve where  $x = 1$ .
- 11) Find the equation of the tangent to the curve  $f(x) = \sin 2x$  at the point on the curve where  $x = \frac{\pi}{2}$ .
- 12) At how many different values of  $x$  does the curve  $y = x^3 - 2x + 1$  have a tangent line parallel to the line  $y = x + 1$ ?
- 13) Find the equation of the tangent to  $f(x) = x^{4x}$  at the point where  $x = 1$ .
- 14) Find the equation of the tangent to  $f(x) = (2x + 1)^{x^2 + 1}$  at the point where  $x = 0$ .

**Answer Key**

1)  $y = 2x - 1$

2)  $y = -\frac{1}{2}x + 4$

3)  $y = -3x - 3$ , Intersections:  $(1, -6), (2, -9)$

4)  $y = 6x - 14$  ;  $y = 6x + 18$

5)  $y = \frac{1}{3}x + 4$

6)  $y = x$ , No additional intersection points.

7)  $y = 4x - 4\pi + 1$

8)  $y = -2x - 1$  ;  $y = -2x + 3$

9)  $y = 2x - e$

10)  $y = x - 1$

11)  $y = -2x + \pi$

12) 2 Points

13)  $y = 4x - 3$

14)  $y = 2x + 1$

## Tangent and Normal Lines – Exercises with a Constant

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### Questions

- 1) Determine the constant  $k$  such that the line  $y = -x + 3$  is tangent to the curve  $y = \frac{k}{x+1}$ .
- 2) Determine the constant  $n$  such that the line  $y = -x + n$  is tangent to the curve  $y = \frac{1}{x}$ .
- 3) Determine the constants  $a, c$  such that the line  $y = ax + 0.5$  is tangent to the curve  $y = \frac{2}{x+c}$  at the point where  $x = 0$ .
- 4) Write the equation of the straight line passing through the point  $(-1, 0)$  and tangent to the curve  $f(x) = \sqrt{x}$ . Sketch the curve and the line.
- 5) Write the equation of the straight lines passing through  $(1.5, 0)$  and tangent to the curve  $f(x) = \frac{1}{4}x + 1$ , Sketch the curve and the lines and prove that the lines are perpendicular.
- 6) Find all lines that can be drawn tangent to the curve  $f(x) = x^2 - 2x + 1$  from the point  $(2, -3)$ . Sketch the curve and the lines.
- 7) Determine the constant  $b$  such that the line  $y = 3x$  is tangent to the curve  $y = x\sqrt{x} + b$ .
- 8) Determine the constant  $c$  such that the curve  $y = -0.5x^2 + c$  is tangent to the curve  $y = \frac{1}{x}$ , Find the tangent point and the joint tangent.
- 9) Determine the constant  $c$  such that the parabola  $y = -x^2 + c$  is tangent to the parabola  $y = x^2 - 4x + 6$ , Find the tangent point and the joint tangent.
- 10) Determine the constant  $c$  such that the curve  $y = -8x^2 + c$  is tangent to the curve  $y = \frac{1+6x^2}{2x^2}$ , Find the tangent point(s) and the joint tangents(s).

**Answer Key**

1)  $k = 4.$

2)  $n = \pm 2.$

3)  $a = -\frac{1}{8}, c = 5.$

4)  $y = \frac{1}{2}x + \frac{1}{2}.$

5)  $y = 2x - 3, y = -\frac{1}{2}x + \frac{3}{4}.$

6)  $y = 6x - 15, y = -2x + 1.$

7)  $b = 4.$

8)  $c = 1.5$ , tangent point:  $(1, 1)$ , joint tangent:  $y = x + 2.$

9)  $c = 4$ , tangent point:  $(1, 3)$ , joint tangent:  $y = -2x + 5.$

10)  $c = 7$ , tangent point:  $\left(\frac{1}{2}, 5\right)$ , joint tangent:  $y = -8x + 9$

$c = 7$ , tangent point:  $\left(-\frac{1}{2}, 5\right)$ , joint tangent:  $y = 8x + 9.$

## Tangent and Normal Lines of Implicit Functions

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### Questions

- 1) Find the equation of the line tangent to  $x^2 + y^2 = 2xy - 2x - y + 6$  at  $(2, 2)$ .
- 2) Find the equation of the line tangent to  $x^3 + 4xy - 4y^3 = 1$ . At the point  $(x, y)$  on the curve where  $x = 1, y > 0$ .
- 3) Find the equation of the line tangent to  $\sqrt{x} + \sqrt{y} = 2\sqrt{a}$ ,  $a > 0$ , at the point on the curve where  $x = a$ .

### Answer Key

1)  $y = -2x + 6$

2)  $y = -\frac{1}{11}x - \frac{10}{11}$

3)  $y = -x + 2a$



## Tangent and Normal Lines – Parametric

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### Questions

- 1) Give the following function  $x = f(t)$ ,  $y = g(t)$  where:  $x = \sqrt{2t^2 + 1}$ ,  $y = 4t^2 + 4t + 1$ :
- Find  $\frac{dy}{dx}$ .
  - Find the equation of the line tangent to the curve at the point for which  $t = 2$ .
- 1) Give the following function  $x = f(t)$ ,  $y = g(t)$  where:  $x = \sqrt{2t^3 + 5t^2}$ ,  $y = \sqrt[3]{4t}$ :
- Find  $\frac{dy}{dx}$ .
  - Find the equation of the line tangent to the curve at the point for which  $t = 2$ .

### Answer Key

- 1) a.  $\frac{dy}{dx} = \frac{(4t+2)(\sqrt{2t^2+1})}{t}$       b.  $y = 15x - 20$ .
- 2) a.  $\frac{dy}{dx} = \frac{4\sqrt{2t^3+4t^2}}{3\sqrt[3]{(4t)^2 \cdot (3t^2+5t)}}$       b.  $y = \frac{2}{\sqrt{22}}x - \frac{22 - 6\sqrt{22}}{11}$ .

## The Angle Between Two Curves

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### Questions

- 1) Show that the curve  $x^2 + 2y^2 = 8$  and the curve  $x^2 - y^2 = 2$  intersect at right angles.
- 2) Find the angles between the following pairs of curves:
  - a.  $y = x^2$ ,  $y = \frac{1}{x}$
  - b.  $x^2 + y^2 = 8$ ,  $y^2 = 2x$

### Answer Key

- 1) Slope 1:  $\frac{-1}{\sqrt{2}}$ , Slope 2:  $\sqrt{2} \cdot \frac{-1}{\sqrt{2}} \cdot \sqrt{2} = -1 \Rightarrow$  The curves are perpendicular
- 2) a.  $\alpha = 71.57^\circ$       b.  $\alpha = 71.57^\circ$

## Vertical Tangents and Cusps

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### Questions

1) Answer the following questions:

- Find all the points on the graph  $y = \sqrt[5]{4-x}$  where the tangent line is vertical.
- Does the function have a vertical cusp?

2) Answer the following questions:

- Find all the points on the graph  $y = \sqrt{x} + \sqrt[3]{x}$  where the tangent line is vertical.
- Does the function have a vertical cusp?

3) Answer the following questions:

- Find all the points on the graph  $y = \sqrt[3]{x^2}$  where the tangent line is vertical.
- Does the function have a vertical cusp?

4) Answer the following questions:

- Find all the points on the graph  $y = |x^3 - 27|$  where the tangent line is vertical.
- Does the function have vertical cusp?

5) Answer the following questions:

- Find all the points on the graph  $y = \sqrt{4-x^2}$  where the tangent line is vertical.
- Does the function have vertical cusp?

6) Answer the following questions:

- Find all the points on the graph  $f(x) = \begin{cases} x^{\frac{1}{3}} + 4 & x \leq 0 \\ 4 - x^{\frac{1}{5}} & x > 0 \end{cases}$  where the tangent line is vertical.
- Does the function have vertical cusp?

**Answer Key**

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|----------------------|--------|
| 1) a. $(4,0)$        | b. No  |
| 2) a. $(2,0),(-2,0)$ | b. No  |
| 3) a. $(0,0)$        | b. Yes |
| 4) a. None           | b. No  |
| 5) a. $(-2,0),(2,0)$ | b. No  |
| 6) a. $(0,4)$        | b. Yes |

## Linear Approximation

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### Questions

- 1) Use linear approximation to approximate the value of  $\sqrt[5]{33}$ .
- 2) Use linear approximation to approximate the value of  $\sqrt[4]{15}$ .
- 3) Use linear approximation to approximate the value of  $\sin 3^\circ$ .
- 4) Use linear approximation to approximate the value of  $\arctan 0.25$ .
- 5) Use linear approximation to approximate the value of  $\frac{1}{e}$ .

### Answer Key

- 1) 2.0125                      2) 1.96875                      3) 0.05236                      4) 0.25  
5) 0