

# Workbook



## Table of Contents

The Limit of a Function .....	2
Technique 1 – Substitution .....	2
Technique 2 – Factoring.....	4
Technique 3 – Multiplication by Conjugate .....	6
Technique 4 – $f(x)$ Tends to Infinity .....	8
Technique 5 – $x$ Tends to Infinity .....	10
Technique 6 – Euler’s Limit .....	8
Technique 7 – Trigonometric Limits .....	9
Technique 8 – The Sandwich/Squeeze Theorem.....	10
Technique 9 – Piecewise Functions .....	11
Limit from Definition.....	12

# The Limit of a Function

## Technique 1 – Substitution

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### Questions

Find the following limits if possible:

1)  $\lim_{x \rightarrow 4} x^2 + x + 1$       2)  $\lim_{x \rightarrow 10} \frac{x+1}{x+2}$       3)  $\lim_{x \rightarrow 1^+} \sqrt{x+3}$       4)  $\lim_{x \rightarrow 100} 20$

### Answer Key

1) 21                      2)  $\frac{11}{12}$                       3) 2                      4) 20

## Technique 2 – Factoring

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### Questions

Find the following limits if possible:

1)  $\lim_{x \rightarrow 3} \frac{x^2 - x - 6}{x^2 - 9}$

2)  $\lim_{x \rightarrow -5} \frac{2x^2 - 50}{2x^2 + 3x - 35}$

3)  $\lim_{x \rightarrow 1} \frac{x^7 - x}{x - 1}$

4)  $\lim_{x \rightarrow 1} \frac{x^n - x}{x - 1} \quad n > 1$

### Answer Key

1)  $\frac{5}{6}$

2)  $\frac{20}{17}$

3) 6

4)  $n - 1$

## Technique 3 – Multiplication by Conjugate

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### Questions

Calculate the following limits if possible:

1)  $\lim_{x \rightarrow 1} \frac{1 - \sqrt{x}}{1 - x}$

2)  $\lim_{x \rightarrow 3} \frac{x - 3}{\sqrt{x + 1} - 2}$

3)  $\lim_{x \rightarrow 3} \frac{3 - \sqrt{x + 6}}{2x - 6}$

4)  $\lim_{x \rightarrow 1} \frac{\sqrt{x^2 + x + 2} - 2}{x^2 - 1}$

5)  $\lim_{x \rightarrow 4} \frac{\sqrt{2x + 1} - \sqrt{x + 5}}{x - 4}$

6)  $\lim_{x \rightarrow 1} \frac{2 - \sqrt{3x + 1}}{1 - \sqrt{x - 1}}$

7)  $\lim_{x \rightarrow 1} \frac{1 - \sqrt[3]{x}}{1 - x}$

### Answer Key

1)  $\frac{1}{2}$

2) 4

3)  $-\frac{1}{12}$

4)  $\frac{3}{8}$

5)  $\frac{1}{6}$

6)  $\frac{3}{4}$

7)  $\frac{1}{3}$

## Technique 4 – $f(x)$ Tends to Infinity

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### Questions

Calculate the following limits if possible:

1)  $\lim_{x \rightarrow 0} \frac{x^2 + 4}{x}$

2)  $\lim_{x \rightarrow 2} \frac{(x-1)^2}{x-2}$

3)  $\lim_{x \rightarrow 2} \frac{x^2 - 1}{(x-2)(x-5)}$

4)  $\lim_{x \rightarrow 0^+} \frac{\ln x}{x}$

5)  $\lim_{x \rightarrow 2^-} -\frac{1}{2} \ln(2-x)$

6)  $\lim_{x \rightarrow 0^+} ((\ln x)^2 + x(\ln x - 3))$

7)  $\lim_{x \rightarrow 0} e^{\frac{1}{x}}$

8)  $\lim_{x \rightarrow 0^+} \frac{1}{1 + 2^{\frac{1}{x}}}$

9)  $\lim_{x \rightarrow 0^-} \frac{1}{1 + 2^{\frac{1}{x}}}$

10)  $\lim_{x \rightarrow 0} \frac{1}{1 + 2^{\frac{1}{x}}}$

11)  $\lim_{x \rightarrow 0^+} \ln x \cdot \cot x$

### Answer Key

- 1) No limit      2) No limit      3) No limit      4)  $-\infty$       5)  $\infty$       6)  $\infty$   
 7) No limit      8) 0      9) 1      10) No limit      11)  $-\infty$

## Technique 5 – $x$ Tends to Infinity

### Questions

Find the following limits if possible:

- 1)  $\lim_{x \rightarrow \infty} (e^{-x})^{\ln x}$
- 2)  $\lim_{x \rightarrow -\infty} \arctan(x) + e^x$
- 3)  $\lim_{x \rightarrow \infty} \frac{4x^2 + 2}{x^2 + 1000x}$
- 4)  $\lim_{x \rightarrow -\infty} \frac{x^4 + 2x^2 + 6}{3x^2 + 10x}$
- 5)  $\lim_{x \rightarrow \infty} \frac{x^4 + 2x^2 + 6}{3x^5 + 10x}$
- 6)  $\lim_{x \rightarrow \infty} \left( \frac{x^2 - 5x + 6}{2x + 10} - \frac{x}{2} \right)$
- 7)  $\lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 1}}{x}$
- 8)  $\lim_{x \rightarrow -\infty} \frac{\sqrt{x^2 + 1}}{x}$
- 9)  $\lim_{x \rightarrow -\infty} \frac{\sqrt{9x^6 - 5x}}{x^3 - 2x^2 + 1}$
- 10)  $\lim_{x \rightarrow \infty} \frac{\sqrt[3]{x^4 + 2x^2 + 6 + 27x^6}}{\sqrt{3x^3 + 10x + 4x^4}}$
- 11)  $\lim_{x \rightarrow \infty} \frac{\sqrt{x+2} - \sqrt{3x-3}}{\sqrt{4x+1} - \sqrt{5x-1}}$
- 12)  $\lim_{x \rightarrow -\infty} \frac{16^x + 4^{x+1}}{2^{4x+2} + 2^{x+3}}$
- 13)  $\lim_{x \rightarrow \infty} \frac{4 \cdot 9^x + 3^{x+1}}{81^{0.5x} + 3^{x+3}}$
- 14)  $\lim_{x \rightarrow -\infty} \frac{4 \cdot 9^x + 3^{x+1}}{81^{0.5x} + 3^{x+3}}$
- 15)  $\lim_{x \rightarrow \infty} \sqrt{\frac{4x^2 + 2}{x^2 + 1000x}}$
- 16)  $\lim_{x \rightarrow \infty} \ln \left( \frac{3x^3 - 5x - 1}{x^3 - 2x^2 + 1} \right)$
- 17)  $\lim_{x \rightarrow \infty} e^{\frac{x^4 + 2x^2 + 6}{3x^4 + 10x}}$
- 18)  $\lim_{x \rightarrow -\infty} \sin \left( \frac{x^4 + 2x^2 + 6}{3x^5 + 10x} \right)$
- 19)  $\lim_{x \rightarrow \infty} (\sqrt{x^2 + kx} - x)$
- 20)  $\lim_{x \rightarrow \infty} (\sqrt{x^2 + x + 1} - x)$
- 21)  $\lim_{x \rightarrow -\infty} (\sqrt{x^2 + x + 1} + x)$
- 22)  $\lim_{x \rightarrow \infty} (\sqrt{x^4 + x^2 + 1} - x^2)$
- 23)  $\lim_{x \rightarrow \infty} (\sqrt{x^2 + ax} - \sqrt{x^2 + bx})$

**Answer Key**

1) 0

2)  $-\frac{\pi}{2}$

3) 4

4)  $-\infty$

5) 0

6) -5

7) 1

8) -1

9) -3

10)  $\frac{3}{2}$

11)  $\frac{1-\sqrt{3}}{2-\sqrt{3}}$

12) 0

13) 4

14)  $\frac{1}{9}$

15) 2

16)  $\ln 3$

17)  $e^{\frac{1}{3}}$

18) 0

19)  $\frac{k}{2}$

20)  $\frac{1}{2}$

21)  $-\frac{1}{2}$

22)  $\frac{1}{2}$

23)  $\frac{a-b}{2}$



## Technique 6 – Euler’s Limit

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### Questions

Find the following limits if possible:

1)  $\lim_{x \rightarrow -\infty} \left(1 + \frac{1}{2x}\right)^x$

2)  $\lim_{x \rightarrow -\infty} \left(1 + \frac{1}{x^2}\right)^x$

3)  $\lim_{x \rightarrow \infty} \left(\frac{x+2}{x}\right)^x$

4)  $\lim_{x \rightarrow \infty} \left(1 - \frac{1}{x^2}\right)^{x^2-1}$

5)  $\lim_{x \rightarrow \infty} \left(\frac{2x+3}{2x-3}\right)^x$

6)  $\lim_{x \rightarrow 0} (1 + \sin x)^{\frac{1}{x}}$

7)  $\lim_{x \rightarrow \infty} \left(\frac{2x+3}{2x-3}\right)^x$

8)  $\lim_{x \rightarrow \infty} \left(\frac{x^2 + x + 1}{x^2 + x + 4}\right)^{4x^2}$

9)  $\lim_{x \rightarrow \infty} \left(1 + \tan \frac{1}{x}\right)^x$

### Answer Key

1)  $\sqrt{e}$

2) 1

3)  $e^2$

4)  $e^{-1}$

5)  $e^3$

6)  $e$

7)  $e^3$

8)  $e^{-12}$

9)  $e$

**Technique 7 – Trigonometric Limits**

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**Questions**

Find the following limits if possible:

1)  $\lim_{x \rightarrow 0} \frac{\sin(3x)}{4x}$

2)  $\lim_{x \rightarrow 0} \frac{\sin(3x)}{\sin(4x)}$

3)  $\lim_{x \rightarrow 0} \frac{x \cos(x)}{\sin(2x)}$

4)  $\lim_{x \rightarrow 0} \frac{1 - \cos(x)}{x^2}$

5)  $\lim_{x \rightarrow 0} \frac{\tan(x) \sin(x)}{x}$

6)  $\lim_{x \rightarrow 0} \frac{\sqrt{1 - \sin(x)} - \sqrt{\cos(x)}}{x}$

7)  $\lim_{x \rightarrow 0} \frac{1 - \cos(1 - \cos(x))}{x^4}$

8)  $\lim_{x \rightarrow 0} \frac{3 \sin(x) - \sin(3x)}{x^3}$

9)  $\lim_{x \rightarrow 0} \frac{1 - \sqrt{\cos(x)}}{x^2}$

**Answer Key**

1)  $\frac{3}{4}$

2)  $\frac{3}{4}$

3)  $\frac{1}{2}$

4)  $\frac{1}{2}$

5)  $\frac{1}{2}$

6)  $\frac{1}{2}$

7)  $\frac{1}{8}$

8) 4

9)  $\frac{1}{4}$

**Technique 8 – The Sandwich/Squeeze Theorem**

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**Questions**

Find the following limits if possible:

1)  $\lim_{x \rightarrow \infty} \frac{\sin x}{x}$

2)  $\lim_{x \rightarrow \infty} \frac{\cos(2x+1)}{x}$

3)  $\lim_{x \rightarrow \infty} \frac{3x + \sin x}{4x + \cos x}$

4)  $\lim_{x \rightarrow \infty} \frac{3x^2 + x + \sin 2x}{x^2 + \cos 3x}$

5)  $\lim_{x \rightarrow 0} x \cdot \sin\left(\frac{1}{x}\right)$

6)  $\lim_{x \rightarrow 0} x^2 \cdot \cos(\ln x^2)$

7)  $\lim_{x \rightarrow \infty} \frac{3x + \arctan(2x - 3)}{4x + \arctan(x - \ln x)}$

8)  $\lim_{x \rightarrow \infty} \sqrt[x]{2^x + 3^x + 4^x}$

9)  $\lim_{x \rightarrow \infty} \frac{1}{x} [x]$

**Answer Key**

1) 0

2) 0

3)  $\frac{3}{4}$

4) 3

5) 0

6) 0

7)  $\frac{3}{4}$

8) 4

9) 1

## Technique 9 – Piecewise Functions

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### Questions

Find the following limits if possible:

$$1) \lim_{x \rightarrow 0} f(x); f(x) = \begin{cases} \frac{\sin 4x}{x} & x > 0 \\ 4 + e^{\frac{1}{x}} & x < 0 \end{cases}$$

$$2) \lim_{x \rightarrow 1} f(x); f(x) = \begin{cases} \frac{x^2 + x - 2}{x - 1} & x > 1 \\ \frac{x - 1}{\sqrt{x} - 1} & x < 1 \end{cases}$$

$$3) \lim_{x \rightarrow 0} \frac{|x|}{x}$$

$$4) \lim_{x \rightarrow \infty} \frac{|x|}{x}$$

$$5) \lim_{x \rightarrow -\infty} \frac{|x|}{x}$$

### Answer Key

- 1) 4
- 2) No limit
- 3) No limit
- 4) 1
- 5) -1

## Limit from Definition

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### Questions

1) Use the definition of the limit to prove that  $\lim_{x \rightarrow 2} (7x + 14) = 28$ .

2) Use the definition of the limit to prove that  $\lim_{x \rightarrow 3} x^2 = 9$ .

3) Use the definition of the limit to prove that  $\lim_{x \rightarrow 1} (x^2 - 1) = 0$ .

4) Use the definition of the limit to prove that  $\lim_{x \rightarrow 24} \sqrt{x+1} = 5$ .

5) Use the definition of the limit to prove that  $\lim_{x \rightarrow 1} \frac{1}{x} = 1$ .

6) Use the definition of the limit to prove that  $\lim_{x \rightarrow \frac{\pi}{4}} \sin x = \sin \frac{\pi}{4}$ .

7) Use the definition of the limit to prove that  $\lim_{x \rightarrow 2} \frac{3+x}{x^2+1} = 1$ .

8) Use the definition of the limit to prove that  $\lim_{x \rightarrow 4^-} (\sqrt{4-x}) = 0$ .

9) Use the definition of the limit to prove that:

a.  $\lim_{x \rightarrow 0^+} \frac{|x|}{x} = 1$

b.  $\lim_{x \rightarrow 0^-} \frac{|x|}{x} = -1$

10) Use the definition of the limit to prove that  $\lim_{x \rightarrow 2} \frac{-5}{(x-2)^2} = -\infty$ .

11) Use the definition of the limit to prove that  $\lim_{x \rightarrow 3^-} \frac{-2}{x-3} = \infty$ .

12) Use the definition of the limit to prove that  $\lim_{x \rightarrow 0^+} \ln x = -\infty$ .

- 13) Use the definition of the limit to prove that  $\lim_{x \rightarrow \infty} \frac{x+7}{x+2} = 1$ .
- 14) Use the definition of the limit to prove that  $\lim_{x \rightarrow \infty} \frac{3-4x}{2x+1} = -2$ .
- 15) Use the definition of the limit to prove that  $\lim_{x \rightarrow \infty} \frac{3x^2-1}{x^2+x+1} = 3$ .
- 16) Given a function  $f(x)$  which satisfies  $\lim_{x \rightarrow \infty} f(x) = -5$ .  
 Prove that there exists an  $M > 0$  such that  $f(x) < -4$  whenever  $x > M$ .
- 17) Given a function  $f(x)$  which satisfies  $\lim_{x \rightarrow -\infty} f(x) = 5$ .  
 Prove that there exists an  $M < 0$  such that  $f(x) > 4$  whenever  $x < M$ .
- 18) Given a positive function  $f(x)$  on the interval  $[a, \infty)$  which satisfies  $\lim_{x \rightarrow \infty} f(x) = 0$ .  
 Prove [using *epsilon-delta*] that  $\lim_{x \rightarrow \infty} \sqrt{f(x)} = 0$ .
- 19) Given the limit:  $\lim_{x \rightarrow \infty} \frac{x^2+2x}{\underbrace{x^2+3x+2}_{f(x)}} = L$  (no need to prove).  
 Find a value for  $M > 0$  such that  $|f(x) - L| < 0.1$  whenever  $x > M$ .
- 20) Use the definition of continuity to prove that  $f(x) = 2x - 3$  is continuous at  $x = 4$ .

**Answer Key:**

- 1-18) and 20) To see the answers to those exercises, please refer to the appropriate videos.  
 19)  $x > 30$