

Workbook



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Matrices

Matrix and Basic Operations on Matrices

Questions

1) For each of the following matrix “calculations”, say whether it’s undefined or give the order [size] of the result:

- | | |
|---|---|
| a. $A_{4 \times 6} + B_{4 \times 6}$ | b. $A_{4 \times 6} \cdot B_{4 \times 6}$ |
| c. $A_{4 \times 6} \cdot C_{6 \times 2} - D_{4 \times 2}$ | d. $A_{4 \times 6} \cdot E_{6 \times 4} - B_{4 \times 6}$ |
| e. $B_{4 \times 6} + A_{4 \times 6} \cdot B_{4 \times 6}$ | f. $E_{6 \times 4} (B_{4 \times 6} + A_{4 \times 6})$ |
| g. $(E_{6 \times 4} + A_{4 \times 6}^T) D_{4 \times 2}$ | h. $E_{6 \times 4}^T \cdot B_{4 \times 6}$ |
| i. $E_{6 \times 4} \cdot A_{4 \times 6} \cdot C_{6 \times 2}$ | j. $E_{6 \times 4} (B_{4 \times 6} - A_{4 \times 6})$ |

2) Solve the matrix equation $\begin{bmatrix} x+2y & 3x-2y \\ 2x-5y & 2x+8y \end{bmatrix} = \begin{bmatrix} 2-2z & 5+z \\ -4-3z & 3-12z \end{bmatrix}$, for x, y, z .

3) Given the matrices $C = \begin{bmatrix} 1 & 4 & 2 \\ 4 & 1 & 5 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 4 & 2 \\ 1 & 0 & -1 \\ 4 & 2 & 10 \end{bmatrix}$, $E = \begin{bmatrix} 4 & 1 & 1 \\ -1 & 0 & 1 \\ 4 & 1 & -1 \end{bmatrix}$. Evaluate:

- | | | |
|------------------------|--------------------|---------|
| a. $E + D$ | b. $E - D + I_3$ | c. $5C$ |
| d. $2D + 4E \cdot I_3$ | e. $2tr(D^2 - 2E)$ | f. |

Given the matrices $A = \begin{bmatrix} 4 & 0 \\ 1 & 2 \\ -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 1 \\ 0 & -2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 4 & 2 \\ 4 & 1 & 5 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 4 & 2 \\ 1 & 0 & -1 \\ 4 & 2 & 10 \end{bmatrix}$.

- | | | |
|---------------|------------------------------------|--------------------------|
| g. $4C^T + A$ | h. $\frac{1}{2}A^T + \frac{1}{4}C$ | i. $I_2 \cdot B \cdot C$ |
| j. $trC^T C$ | k. $D \cdot A \cdot B \cdot C$ | l. |

4) Rewrite each of the following SLEs in matrix form $A\underline{x} = \underline{b}$:

$$\text{a. } \begin{cases} 2x + y - z = 3 \\ x + 2y - 4z = 5 \\ 6x + 4y + z = 2 \end{cases}$$

$$\text{b. } \begin{cases} 2x - 3y + z + t = 1 \\ 4x + y + 2z = 4 \\ y + z + t = 1 \\ x - 4z - 2y = 10 \end{cases}$$

5) Given: $A = \begin{bmatrix} 4 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & -6 & 3 \end{bmatrix}$ $\underline{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\underline{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Express each of the following as an SLE:

a. $A\underline{x} = \underline{b}$

b. $A\underline{x} = 4\underline{x} + \underline{b}$

c. $A\underline{x} = -k\underline{x} + \underline{b}$

d. $A\underline{x} = \underline{x}$

e. $A^T \underline{x} = 2\underline{x} + 3\underline{b}$

f.

6) Recall that a square matrix A is *symmetric* if $A^T = A$ and *antisymmetric* if $A^T = -A$.

a. Given a square matrix A , which of the following must be true?

i. $A \cdot A^T$ is symmetric

ii. $A + A^T$ is symmetric

iii. $A - A^T$ is antisymmetric

b. Given 2 antisymmetric $n \times n$ matrices A and B , which of the following must be true:

i. $BABABA$ is antisymmetric

ii. $A^2 - B^2$ is symmetric

iii. $A^2 + B$ is symmetric

c. Given 2 antisymmetric $n \times n$ matrices A and B , which satisfy $AB = -BA$.

Which of the following must be true:

i. AB^3 is antisymmetric

ii. AB^2 is symmetric

iii. $(A - B)^2$ is symmetric

d. Given 2 antisymmetric $n \times n$ matrices A and B , which satisfy $AB = BA$. Prove:

i. AB is antisymmetric

ii. $AB + B$ is antisymmetric

e. Given that the $n \times n$ matrices A , B , AB are symmetric, prove that $A^4 B^4 = B^4 A^4$.

Answer Key

- 1) a. $C_{4 \times 6}$ b. Undefined. c. $B_{4 \times 2}$ d-e. Undefined.
 f. $C_{6 \times 6}$ g. $B_{6 \times 6}$ h. Undefined. i. $B_{6 \times 2}$
 j. $C_{6 \times 6}$

2) $x = 2, y = 1, z = -1$

3) a. $\begin{bmatrix} 5 & 5 & 3 \\ 0 & 0 & 0 \\ 8 & 3 & 9 \end{bmatrix}_{3 \times 3}$ b. $\begin{bmatrix} 4 & -3 & -1 \\ -2 & 1 & 2 \\ 0 & -1 & -10 \end{bmatrix}_{3 \times 3}$ c. $\begin{bmatrix} 5 & 20 & 10 \\ 20 & 5 & 25 \end{bmatrix}$ d. $\begin{bmatrix} 18 & 12 & 8 \\ -2 & 0 & 2 \\ 24 & 8 & 16 \end{bmatrix}$

e. 230 f. $\begin{bmatrix} 8 & 16 \\ 17 & 6 \\ 7 & 21 \end{bmatrix}_{3 \times 2}$ g. $\begin{bmatrix} 2\frac{1}{4} & 1\frac{1}{2} & 0 \\ 1 & 1\frac{1}{4} & 1\frac{3}{4} \end{bmatrix}$ h. $\begin{bmatrix} 8 & 17 & 13 \\ -8 & -2 & -10 \end{bmatrix}_{2 \times 3}$

i. 63 j. $\begin{bmatrix} -32 & 82 & -22 \\ 48 & 87 & 75 \\ -48 & 108 & -36 \end{bmatrix}$

4) a. $\underbrace{\begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -4 \\ 6 & 4 & 1 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_x = \underbrace{\begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}}_b$

b. $\underbrace{\begin{bmatrix} 2 & -3 & 1 & 1 \\ 4 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & -2 & -4 & 0 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}}_x = \underbrace{\begin{bmatrix} 1 \\ 4 \\ 1 \\ 10 \end{bmatrix}}_b$

5) a. $\begin{cases} 4x - 2y + 4z = 1 \\ x - y + z = 2 \\ x - 6y + 3z = 3 \end{cases}$

b. $\begin{cases} -2y + 4z = 1 \\ x - 5y + z = 2 \\ x - 6y - z = 3 \end{cases}$

c. $\begin{cases} (4+k)x - 2y + 4z = 1 \\ x + (-1+k)y + z = 2 \\ x - 6y + (3+k)z = 3 \end{cases}$

d. $\begin{cases} 3x - 2y + 4z = 0 \\ x - 2y + z = 0 \\ x - 6y + 2z = 0 \end{cases}$

e. $\begin{cases} 2x + y + z = 3 \\ -2x - 3y - 6z = 6 \\ 4x + y + z = 9 \end{cases}$

6) a. All three are true.

b. (ii) Is true.

c. All three are true.

d-e. Solution in the recordings.

Matrix Inverse and its Applications

Questions

Find the inverse of the matrix:

1) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

2) $A = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$

3) $A = \begin{bmatrix} 4 & 1.5 \\ 2 & 1 \end{bmatrix}$

4) $A = \begin{bmatrix} 1 & 0 & 2 \\ 4 & -1 & 8 \\ 2 & 1 & 3 \end{bmatrix}$

5) $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & -1 \\ 5 & 2 & 3 \end{bmatrix}$

6) $A = \begin{bmatrix} 2 & -1 & 1 \\ 3 & -2 & 2 \\ 5 & -3 & 4 \end{bmatrix}$

7) $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix}$

Find the values of k for which the following matrixes are invertible:

8) $\begin{bmatrix} 1 & -1 & 1 \\ 5 & -7 & k^2 + 3 \\ 3 & -1 & k + 3 \end{bmatrix}$

9) $\begin{bmatrix} 1 & 1 & 1 & 1 & k \\ 1 & 1 & 1 & k & 1 \\ 1 & 1 & k & 1 & 1 \\ 1 & k & 1 & 1 & 1 \\ k & 1 & 1 & 1 & 1 \end{bmatrix}$

10) Solve the following SLE by using the Inverse Matrix method: $\begin{cases} 2x - y + z = 3 \\ 3x - 2y + 2z = 5 \\ 5x - 3y + 4z = 11 \end{cases}$

11) Solve the following SLE by using the Inverse Matrix method: $\begin{cases} x + 4y + 2z + 4t = 1 \\ x + 2y - z = 0 \\ y + z + t = 1 \\ x + 3y - z - 2t = 0 \end{cases}$

Answer Key

1) $\begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$

2) $\begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$

3) $\begin{bmatrix} 1 & -1.5 \\ -2 & 4 \end{bmatrix}$

4) $\begin{bmatrix} -11 & 2 & 2 \\ 4 & -1 & 0 \\ 6 & -1 & -1 \end{bmatrix}$

5) $\begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ -10 & 1 & 4 \end{bmatrix}$

6) $\begin{bmatrix} 2 & -1 & 0 \\ 2 & -3 & 1 \\ -1 & -1 & 1 \end{bmatrix}$

7) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{4} & \frac{1}{4} \end{bmatrix}$

8) $k \neq -2, k \neq 1$

9) $k = 1, k = -4$

10) $x = 1, y = 2, z = 3$

11) $x = -13, y = 42, z = -5, t = 2$

Properties of the Matrix Inverse

Questions

1) Assume that all the matrices are invertible and of order n , and extract X :

a. $AXC = D$

b. $A^{-1}XC = A^{-1}DC$

c. $P^{-1}X^T P = A$

d. $C^{-1}(A+X)D^{-2} = I$

e. $(A-AX)^{-1} = X^{-1}C$

f. $ABC^T X^{-1}BA^T C = AB^T$

2) Given that $B = \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}$ and that $B^2 X (2B)^{-1} = B + I$. Find X . All matrices are 2×2 .

3) Given that $B^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 4 & -1 & 8 \\ 2 & 1 & 3 \end{bmatrix}$ and that $BYB^T = B^{-1} + B$. Find Y . All matrices are 3×3 .

4) Given that $A^{-1} = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$ and that $5A^T B(I+2A)^{-2} = (7A)^{-2}$. Find B . All matrices are 2×2 .

5) Prove the following:

a. Given: A is a square matrix satisfying $A^2 - 5A - 2I = 0$.

Prove that A is invertible and express A^{-1} in terms of A and I .

b. Given: A is a square matrix satisfying $(A - 3I)(A + 2I) = 0$.

Prove that A is invertible and express A^{-1} in terms of A and I .

6) Prove the following:

a. Given: A is a square matrix satisfying $A^2 - 5A - 2I = 0$.

Prove that A is invertible and express A^{-1} in terms of A and I .

b. Given: A is a square matrix satisfying $(A - 3I)(A + 2I) = 0$.

Prove that A is invertible and express A^{-1} in terms of A and I .

Matrices

7) Given: $A = \begin{bmatrix} -1 & 3 & 0 \\ 3 & -1 & 0 \\ -2 & -2 & 6 \end{bmatrix}$, $p(x) = x^3 - 4x^2 - 20x + 48$.

- Compute $p(A)$.
- Using part a. above, prove that A is invertible and express A^{-1} in terms of A and I .

8) Given: A is a square matrix satisfying $A^4 = 0$.

- Prove that A is noninvertible.
- Prove that the matrix $I - A$ is invertible and find its inverse.

9) Given: $\begin{cases} P^{-1}AP = B \\ Q^{-1}BQ = C \end{cases}$. Prove that there exists an invertible matrix D such that $D^{-1}AD = C$.

Answer Key

1) a. $X = A^{-1}DC^{-1}$

b. $X = D$

c. $X = (P^{-1})^T A^T P^T$

d. $X = CD^2 - A$

e. $X = (A + C^{-1})^{-1} A$

f. $X = BA^T C \cdot (B^T)^{-1} BC^T$

2) $X = \begin{bmatrix} 20 & -4 \\ -8 & 4 \end{bmatrix}$

3) $Y = \begin{bmatrix} 22 & 86 & 38 \\ 64 & 246 & 114 \\ 56 & 222 & 92 \end{bmatrix}$

4) $B = \frac{1}{245} \begin{bmatrix} 264 & 450 \\ 448 & 768 \end{bmatrix}$

5) Solution in the video.

6) Solution in the video.

7) a. $P(A) = 0$

b. Solution in the video.

8) Solution in the video.

9) Solution in the video.

Elementary Matrices and LU Decomposition

Questions

1) Write the LU decomposition of the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 4 & -1 & 8 \\ 2 & 1 & 6 \end{bmatrix}$.

2) Write the LU decomposition of the matrix $A = \begin{bmatrix} 1 & 2 & -3 & 4 \\ 2 & 3 & -8 & 5 \\ 1 & 3 & 1 & 3 \\ 3 & 8 & -1 & 13 \end{bmatrix}$.

*For the solution see the video.