proprep

Workbook

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Combinatorics

The Multiplication Principle

Questions

1) Calculate the number of possibilities for the following trials:
   a. Throwing a die twice.
   b. A three-digit number.
   c. Selecting a boy and girl from a class with 7 boys and 10 girls.
   d. Distributing 2 different prizes to 10 different people, when each person cannot receive more than one prize.

2) A restaurant offers a business meal that consists of a first course, a main course, and a beverage. The options for the first course are a vegetable salad, a pasta salad, and the soup of the day. The options for the main course are: steak, chicken, meat lasagna, and vegetarian lasagna. The beverage options are: coffee, tea, and lemonade.
   a. How many different meals can be assembled with this menu?
   b. A person orders a meal randomly. Calculate the following probabilities:
      i. A meal with a vegetable salad, meat lasagna, and lemonade.
      ii. A meal with a salad, lasagna, and tea.

3) A five-digit number is selected randomly. Calculate the following probabilities:
   a. The number is even.
   b. All the digits in the number are different.
   c. All the digits in the number are the same.
   d. The number has at least two different digits.
   e. The number has at least two identical digits.
   f. The number is a palindrome.
   (a number that is the same from right to left and from left to right)

4) Five random people enter an elevator in a building with eight floors.
Calculate the following probabilities:
   a. All the people got off at the fifth floor.
   b. All the people got off at the same floor.
   c. All the people got off at different floors.
   d. Mike and Danny got off at the sixth floor, and the other people got off at the other floors.

5) A political party has 15 members.
Three of the members are selected for three different positions.
In how many ways can the positions be distributed, if:

a. A member can serve in more than one position.
b. A member cannot serve in more than one position.

6) A die is thrown four Times.

a. What is the probability that all four results are the same?
b. What is the probability that all the results are different?
c. What is the probability that at least two results are the same?
d. What is the probability that at least two results are different?

7) Spell a five-letter word from the alphabet (26 letters) that does not necessarily have any meaning.

a. What is the probability that there is no D, A, or L in the word?
b. What is the probability that all the letters in the word are identical?
c. What is the probability that at least two letters in the word are different from each other?
d. What is the probability that the word is a palindrome
(a word that is the same whether read right to left or left to right)?

8) A code with 'a' digits is created (each digit can be used more than once in the code).
Calculate the following probabilities (express your answers using 'a').

a. The code does not contain the digit 5.
b. The code contains the digit 3.
c. The code contains no odd digits.

9) In a game of chance, a form with n cells is filled out.
Each cell is marked with a ✓ or an ✗.
In how many different ways can the form for the game be filled out?

Answer Key

1) a. 36 b. 900 c. 70 d. 90

2) a. 36 b.i. $\frac{1}{36}$ b.ii. $\frac{1}{9}$

3) a. 0.5 b. 0.3024 c. 0.0001 d. 0.9999 e. 0.6976 f. 0.01

4) a. 0.00003 b. 0.00024 c. 0.20508 d. 0.01047

5) a. 3,375 b. 2,730

6) a. $\frac{1}{216}$ b. $\frac{5}{18}$ c. $\frac{13}{18}$ d. $\frac{215}{216}$

7) a. $\frac{23^5}{26^5}$ b. $\frac{1}{26^4}$ c. $1-\frac{1}{26^4}$ d. $\frac{1}{26^2}$

8) a. 0.9^a b. $1-0.9^a$ c. 0.5^a

9) 2^n

Permutations – Objects Ordered in a Row

Questions

1) Calculate the following:
   a. In how many ways can four books be ordered on a shelf?
   b. In how many ways can five soldiers stand in line

2) 10 disks have been ordered randomly on a shelf, 2 of which are in French.
   a. What is the probability that the French disks are next to each other?
   b. What is the probability that the French disks are not next to each other?
   c. What is the probability that the 2 French disks are on opposite ends of the shelf?

3) Five boys and four girls wrote an exam and then ranked according to their marks. Assume that no students have the same mark.
   a. How many rankings are possible?
   b. How many rankings are possible if boys and girls are marked separately?

4) 10 different books are ordered on the shelf.
   a. In how many different ways can the books be ordered on the shelf?
   b. 2 out of the 10 books are about statistics.
      What is the probability that if the books are put in random order, the statistics books will be next to each other?
   c. What is the probability that if the books are put in random order, the statistics books will not be next to each other?
   d. What is the probability that the statistics books are on the edges of the shelf (each book at a different end)?

5) A person creates a playlist with twelve different songs.
Four of the songs are in Spanish, five are in English, and three are in French.
The person plays the songs randomly from the playlist.
   a. What is the probability that all the English songs appear as a single group at the beginning of the list?
   b. What is the probability that all the English songs appear consecutively?
   c. What is the probability that songs in the same language appear consecutively (all the songs in English consecutively, all the songs in Spanish consecutively, and all the songs in French consecutively)?

6) Four boys and four girls sit randomly in a row in a movie theater in seats 1-8.

a. What is the probability that John and Monica did not sit next to each other?
b. What is the probability that the boys sat in the odd-numbered seats?
c. What is the probability that all the boys sat next to each other?
d. What is the probability that the boys sat next to each other and the girls sat next to each other?

Answer Key

1) a. 24 b. 120

2) a. 0.2 b. 0.8 c. $\frac{1}{45}$

3) a. 362,880 b. 2,880

4) a. 3,628,880 b. 0.2 c. 0.8 d. $0.022 = \frac{1}{45}$

5) a. $\frac{1}{792}$ b. $\frac{1}{99}$ c. $\frac{1}{4620}$

6) a. 0.75 b. 0.0143 c. 0.0714 d. 0.0286

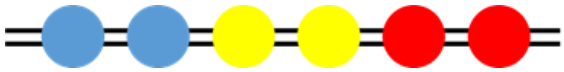
Permutations with Identical Objects

Questions

1) What is the number of permutations in coloring 2 out of 5 cells?

2) In how many different ways can the letters *b, m, m, b, m, c* be ordered?

3) Six lights are mounted on a ceiling rail – two blue, two yellow and two red.
In how many ways can the lights be ordered?



4) How many numbers can be formed out of the following digits: $1,\ 2,\ 2,\ 2,$ and 6 ?

5) In a game of chance, there are 2 Players and 10 squares. Player 1 colors four squares, and Player 2 needs to guess which four squares were colored.
What is the probability that Player 2 will guess all four squares in one try?

6) How many different signals, each composed of ten flags arranged in a row, can be created, if four of the flags are white, three are blue, two are red, and one is black? (Assume flags with the same color are completely identical)

Answer Key

1) 10
2) 60
3) 90
4) 20
5) $\frac{1}{210}$
6) 12,600

Permutations with and without Returns

Questions

1) In a political party with twenty members, it is necessary to select three members for three different positions.

a. A member can serve in more than one position. How many combinations are there for distributing the positions?

b. A member can serve in only one position. How many combinations are there for distributing the positions?

2) In a game of chance, there are four boxes marked A to D.
Each box must be filled with a digit (0-9).
The winner is the one who guesses all the digits in the all the boxes correctly.

a. What is the probability of winning the game?

b. What is the probability that the digit 3 does not appear in any box in the winning number?

c. What is the probability that the digit 4 appears at least once in the winning number?

3) There are 22 people in a group.
What is the probability that at least two of them have the same birthday?

4) Three people arrange to meet each other at the Hilton Hotel in Singapore.
Unfortunately, there are five Hilton Hotels in Singapore.

a. What is the probability of all three people meeting at the same hotel?

b. What is the probability of all three people going to different hotels?

5) There are 40 students in a class. 5 must be chosen for a class committee.
In how many different ways can the committee be selected if:

a. The committee has five different positions, and a student can serve in more than one position.

b. The committee has five different positions, and a student cannot serve in more than one position.

Answer Key

1) a. 8,000 b. 6,840

2) a. $\frac{1}{10,000}$ b. $\frac{9^4}{10^4}$ c. $1-\frac{9^4}{10^4}$

3) 0.476

4) a. $\frac{1}{25}$ b. 0.48

5) a. 40^5 b. $(40)_5 = 78,960,960$

Non-Sequential Sampling without Returns

Questions

1) There are 15 girls and 10 boys in a class.
Five different students are chosen for a committee.
How many possible ways can the committee be composed, if:

a. The choice of boys and girls in the committee is unrestricted.
b. The committee must consist of three girls and two boys.
c. There must be no boys in the committee.

2) A student wishes to select five elective courses in the semester.
There are 10 available courses: 5 humanities courses, 3 social sciences courses and 2 mathematics courses.

a. How many different selections can he create for himself?
b. How many selections include three humanities courses?
c. How many selections have two courses that are **not** humanities courses?
d. How many selections have two humanities courses, two social sciences courses, and one mathematics course?

3) There are 30 students in a class, of which 12 are males and 18 are females.
A committee of four students must be chosen from the class.
The students are chosen randomly.

a. What is the probability that the committee will consist solely of females?
b. What is the probability that the committee will have only one female?
c. What is the probability that the committee will have at least one female?

4) In a lottery, five numbers must be selected from 45.
The numbers are 1-45.

a. What is the probability that the winning entry will have all even numbers?
b. What is the probability that the winning entry will have at most one even number?
c. What is the probability that the winning entry will have at least one even number?
d. What is the probability that the winning entry will consist solely of numbers larger than 30?

5) There are 52 cards in a pack: 13 black spades, 13 red hearts, 13 red diamonds, and 13 black clubs. Each of the four suits has nine cards numbered 2-10, and the remaining cards are a jack, a queen, a king, and an ace (the deck is an ordinary one with no jokers). Two people are playing poker.
Each of them is dealt five cards randomly (with no returns).

a. What is the probability that the first player will get all the kings and the second will get all the queens?
b. What is the probability that one of the players will get the ace of hearts?
c. What is the probability that the first player will get all black cards and the second player will get exactly two black cards?
d. What is the probability that the first player will get at least three cards with a number (an ace is not a number)?

6) There are four study tracks at a college, and five secretaries for each study track.
A five-member committee is selected randomly from all the secretaries at the college.
Calculate the following probabilities:

a. All the secretaries on the committee are from one specific study track.
b. All the secretaries on the committee are from the same study track.
c. At least one secretary is selected from each study track.

7) Prove that $\binom{n}{k}+\binom{n}{k+1}=\binom{n+1}{k+1}$.

8) $2n$ boys and $2n$ girls are divided into two groups.

a. In how many different ways can the groups be selected if the two groups must be the same size, and each group has the same number of girls and boys?
b. In how many different ways can the groups be selected if each group has an equal number of girls and boys, but the two groups are not necessarily the same size?

Answer Key

1) a. 53,130 b. 20,475 c. 3,003

2) a. 252 b. 100 c. 100 d. 60

3) a. 0.1117 b. 0.1445 c. 0.9819

4) a. 0.0216 b. 0.187 c. 0.972 d. 0.00246

5) a. $\dfrac{\binom{4}{4}\binom{44}{1}\binom{4}{4}\binom{43}{1}}{\binom{52}{5}\binom{47}{5}} \approx 0$ b. 0.1923 c. 0.009 d. 0.837

6) a. $\dfrac{1}{15504} = 6.45 \cdot 10^{-5}$ b. $\dfrac{4}{15504}$ c. 0.3225

7) $A = \binom{n+1}{k+1} = \dfrac{(n+1)!}{(k+1)!(n+1(k+1))!} = \dfrac{(n+1)!}{(k+1)!(n-k)!} = A$

$$B = \binom{n}{k} + \binom{n}{k+1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-(k+1))!} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-k-1)!}$$

$$= \frac{n!(k+1)}{k!(k+1)(n-k)!} + \frac{n!(n-k)}{(k+1)!(n-k-1)!(n-k)} = \frac{n!(k+1)}{(k+1)!(n-k)!} + \frac{n!(n-k)}{(k+1)!(n-k)!}$$

$$= \frac{n!(k+1)+n!(n-k)}{(k+1)!(n-k)!} = \frac{n!(k+1+n-k)}{(k+1)!(n-k)!} = \frac{n!(n+1)}{(k+1)!(n-k)!} = \frac{(n+1)!}{(k+1)!(n-k)!} = A$$

8) a. $\left[\binom{2n}{n}\right]^2$ b. $\sum_{i=1}^{n}\left[\binom{2n}{i}\right]^2$

Non-Sequential Sampling with Returns

Questions

1) In how many ways can eight identical balls be put into five boxes, when a box can contain more than one ball?

2) In how many ways can five identical notebooks be put into three different files?

3) In how many ways can eight balls be put into three different boxes when:
   a. The balls are identical.
   b. The balls are different from each other.

4) What is the number of nonnegative whole numbers solutions to the following equation: $X_1 + X_2 = 3$.

5) What is the number of nonnegative whole numbers solutions to the following equation: $X_1 + X_2 + X_3 + X_4 = 20$.

6) In an auction, four identical gold candlesticks are displayed. Three collectors competed to purchase them. A collector can purchase more than one candlestick.
Assuming that all the candlesticks were sold, how many possible ways could they be sold to the collectors?

7) Two letters are selected from the set of letters $\{A, B, C, D\}$. A letter can be selected more than once, but the order of the selected letters doesn't matter.
How many ways are there of choosing the letters?

8) In the lottery, four numbers must be selected from numbers 1-20.
The order in which the numbers are selected is unimportant.
What are the chances of guessing all four numbers, if:
   a. The same number cannot be selected more than once.
   b. The same number can be selected more than once.

9) Five balls are put into six boxes.
Calculate the number of possible ways of putting the balls into the boxes, where:
   a. The balls are different from each other and a box can contain more than one ball.
   b. The balls are identical, and a box can contain more than one ball.
   c. The balls are different from each other and a box cannot contain more than one ball.
   d. The balls are identical, and a box cannot contain more than one ball.

10) There are k balls to put into n boxes $(n > k)$.

Calculate the number of possible ways of putting the balls in, when:

a. The balls are different from each other and a box can contain more than one ball.
b. The balls are identical, and a box can contain more than one ball.
c. The balls are different from each other and a box cannot contain more than one ball.
d. The balls are identical, and a box cannot contain more than one ball.

Answer Key

1) 495

2) 21

3) a. 45 b. 6,561

4) 4

5) 1,771

6) 15

7) 10

8) a. $\frac{1}{4,850}$ b. $\frac{1}{8,855}$

9) a. $n^k = 6^5 = 7,776$ b. 252 c. 720 d. 6

10)

Sampling Order / Sampling Type	Sequential Sampling *Order* DOES *Matters* (Permutations)	Non Sequential Sampling *Order* DOES NOT *Matter* (Combinations)
WITH Returns	a. n^k	b. $\binom{n+k-1}{k} = \binom{n+k-1}{n-1}$
WITHOUT Returns	c. $(n)_k = \frac{n!}{(n-k)!}$	d. $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Summary Questions

Questions

1) There are 40 students in a class.
A class committee with five positions is selected.
In how many ways can the committee be selected, if:

a. The committee has five different positions, and a student can serve in more than one position.
b. The committee has five different positions, and a student cannot serve in more than one positions.
c. There are no different positions on the committee.

2) There are 30 employees in an office. An overseas delegation with four positions is selected. In how many ways can the delegation be selected, if:

a. The delegation has four different tasks to carry out, and each employee can do more than one task.
b. The delegation has four different tasks to do, and each employee cannot do more than one task.
c. Four different employees will be chosen for a delegation in which everyone has the same position.

3) A secret code is being created. The code is composed of two different digits and three different English letters (26 possible letters).

a. How many different codes can be created?
b. How many different codes can be created if the code begins with a digit and ends with a digit?
c. How many codes can be created if the digits must be next to each other?
d. In how many codes do the digits not appear consecutively?

4) There are four drawers in a closet.
A mother asks her child to arrange six different games in the closet.
The child puts the games randomly into the drawers.
Each drawer can contain all of the games together.

a. What is the probability that the child will put all the games into the top drawer?
b. What is the probability that the child will put all the games into the same drawer?
c. What is the probability that the game of Chess will be put into the uppermost drawer, and the other games will be put into the other drawers?
d. What is the probability that the game of Chess will not be put into the uppermost drawer?

5) In a given city, four political parties – A, B, C and D – are running for the city council. Six people do not know for whom to vote, so they choose a party randomly.

a. What is the probability that all six people will vote for the same party?
b. What is the probability that Party A will get no votes?
c. What is the probability that Party A will receive exactly three votes, and the other parties will get only one vote each?
d. What is the probability that Party A will get two votes, Party B will get two votes, and Party C will get two votes?

6) Five friends meet at the theater to see a movie.
There are eight different movies showing. Each person selects a movie randomly.

a. What is the probability that everyone will select the same movie?
b. What is the probability that everyone will choose the movie "Alien"?
c. What is the probability that each friend will select a different movie?
d. What is the probability that at least two friends will select the same movie?
e. What is the probability that Danny and Eliot will select the movie "Alien" and all the others will select other movies?
f. What is the probability that none of the friends will select the movie "Alien"?
g. All the eight movies are presented to the friends as a list. Three of the eight movies are horror movies. What is the probability that the three horror movies will appear consecutively on the list?

7) There are 10 people in a group.
Two different committees must be assembled from the group: one with four people, and the other with three people.
Each person can be selected to only one committee. Calculate the number of different ways of assembling these committees when:

a. There are no different positions on the committees.
b. Each committee has a leader.
c. Each committee has different positions.

8) Four men and three women sit in one row containing 10 chairs.
How many ways can they be seated, if:

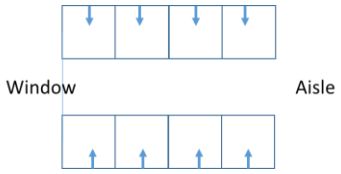
a. Anyone can sit anywhere they want.
b. All the men are seated together, and all the women are seated together.
c. Two men are seated on one end of the row, and the other two men are seated on the other end of the row.

9) There are 10 numbers from 1-10 in a lottery.
Five numbers are chosen randomly.
What is the probability that the number 7 is the second largest of the numbers selected?

10) Six people get on a bus that has 10 stops.
Each person independently and randomly selects at which stop to get off.

a. What is the probability that each person gets off at a different station?
b. What is the probability that exactly three people get off at the fifth stop?
c. What is the probability that Sharon is the only passenger to get off at the second stop?
d. What is the probability that all the passengers get off at the fifth and sixth stations, with at least one person getting off at each of these stops?

11) In a train, there are 4 seats facing forwards and 4 seats facing backwards. Four couples sit randomly in the seats.



*** This diagram is relevant for all sections**

a. In how many different ways can the 4 couples (8 people) sit in the seats?
b. What is the probability that the Smiths sit beside each other facing forward?
c. What is the probability that the Smiths sit beside each other?
d. What is the probability that each of the Smiths has a window seat?
e. What is the probability that each of the Smiths sit where one faces forwards and the other faces backwards?
f. What is the probability that the Smiths sit facing each other (sit opposite one another)?
g. What is the probability that all the men sit facing forwards and all the women sit facing backwards?
h. What is the probability that each couple sit facing each other?

12) A password is composed of 5 characters, where each character can be either a number from 0-9 or a letter in the English alphabet.
Each character may repeat itself more than once.

a. How many different passwords can be created?
b. How many passwords are there where all the characters (letters and numbers) are different?
c. How many passwords have at least one number and one letter?

13) A committee of 3 is chosen out of n different people.
How many different ways can the committee be chosen? Give your answer in terms of n.

a. There are no defined roles in the committee and 3 different people are chosen.
b. There are 3 defined roles in the committee and each person can serve in only one role.
c. There are 3 defined roles in the committee and each person can serve in more than one role.

14) Two people each toss a coin. What is the probability that after tossing the coin n times each, that both people will have the same number of heads?
Express the answer in terms of n.

15) A code has 'a' numbers (the numbers may be repeated).
Calculate the following probabilities and express the answers in terms of 'a':

a. The code does not contain the number 5.
b. The code contains the number 3.
c. The code does not contain odd numbers.

16) How many times should a pair of dice be rolled so that we have a probability of at least 0.5 of getting at least one roll with a sum of 12?

17) A six-digit number is randomly chosen.

a. What is the probability that the number 5 will appear only once?
b. What is the probability that the number 4 will appear at least once and that the number 0 will also appear at least once?

18) Jane has 3 red binders and 2 blue binders on a shelf at her office.
She wrote two notes and placed them randomly between the 5 binders.
(each note may be placed in one of four different places between the binders)

a. What are the chances that the two notes will be placed in different places?
b. What are the chances that between the two notes there are 2 red binders and no blue binders?
c. What are the chances that between the two notes there is exactly one binder?
d. What are the chances that between the two notes there are two binders where one binder is blue?

19) Six pens are placed randomly in three different drawers.
A drawer is chosen for each of the pens.

a. What are the chances that in exactly two drawers each there are exactly two pens?
b. What are the chances that in exactly one drawer there are exactly two pens?
c. What are the chances that in exactly three drawers each there are exactly two pens?

20) n balls are randomly placed in n different boxes, where a box may contain more than one ball. What are the chances that in the i^{th} box $(1 \le i \le n)$ there will be exactly k balls?

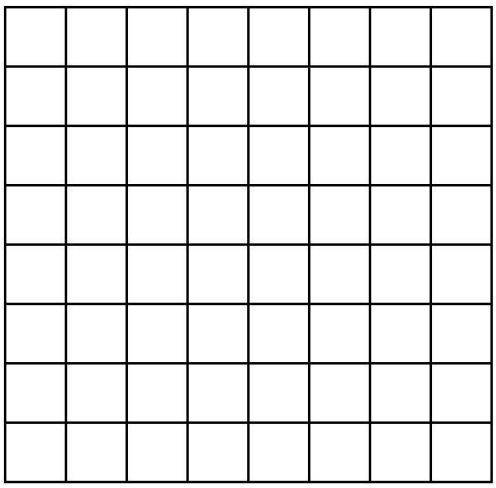
21) There are 6 finalists in a race. Only the top three finalists receive medals. Assume that all finalists finish the race.

a. What is the total number of combinations for the finalists to complete the race?
b. What are the number of combinations where finalist #6 receives a medal?
c. What are the number of combinations where finalist #6 receives a medal or finalist $\#2$ receives the gold medal?

22) A fair die is rolled k times.

a. What are the odds that the highest value is j?
b. What are the odds that the lowest value is i?
c. What are the odds that the highest value is j and the lowest value is i where $i \leq j$?

23) Consider an 8x8 square-tiled board. If two squares on this board are selected randomly, what is the probability that they share a common boundary?



Answer Key

1) a. 40^5 b. $\frac{40!}{35!}=40\times39\times38\times37\times36$ c. $\frac{40!}{5!35!}=658,008$

2) a. 810,000 b. 657,720 c. 27,405

3) a. 14,040,000 b. 1,404,000 c. 5,616,000 d.8,424,000

4) a. $\frac{1}{4096}$ b. $\frac{4}{4096}=0.00098$ c. $\frac{1\times3^5}{4^6}=0.05933$ d. $\frac{3}{4}=0.75$

5) a. $\frac{4}{4096}=0.00098$ b. $\frac{3^6}{4096}=0.17798$ c. $\frac{20\times3\times2\times1}{4096}=0.02929$ d. 0.02197

6) a. $\frac{1}{4096}$ b. $\frac{1}{8^5}=\frac{1}{32,768}$ c. $\frac{(8)_5}{8^5}=0.205$ d. 0.795

e. 0.0105 f. 0.5129 g. 0.1071

7) a. 4,200 b. 50,400 c. 604,800

8) a. 604,800 b. 2,880 c. 2,880

9) 0.238

10) a. 0.1512 b. 0.0146 c. 0.059 d. $\frac{62}{10^6}$

11) a. 4,320 b. 0.1071 c. 0.2142 d. 0.0357

e. 0.5714 f. 0.1429 g. 0.01429 h. 0.0095

12) a. $n^k=36^5$ b. $(n)_k=(36)_5=36\times35\times34\times33\times32$ c. $36^5-\left[10^5+26^5\right]$

13) a. $\frac{n!}{3!(n-3)!}$ b. $(n)_3=n(n-1)(n-2)$ c. n^3

14) $\frac{\sum_{i=0}^{n}\binom{n}{i}^2}{4^n}$

15) a. 0.9^a b. $1-0.9^a$ c. 0.5^a

16) $n\geq\left|24.6\right|\geq25$

17) a. 0.35721 b. 0.1759

18) a. $\frac{3}{4}=0.75$ b. 0.075 c. 0.375 d. 0.15

19) a. 0 b. $\frac{450}{729}$ c. $\frac{90}{729}$

20) $\frac{\binom{n}{k} \cdot (n-1)^{n-k}}{n^n}$

21) a. 720 b. 360 c. 432

22) a. $\frac{j^k - (j-1)^k}{6^k}$ b. $\frac{(7-i)^k - (6-i)^k}{6^k}$ c. $\frac{(j-i+1)^k - (j-i)^k - (j-i)^k + (j-i-1)^k}{6^k}$

23) $\frac{1}{18}$