proprep

Workbook

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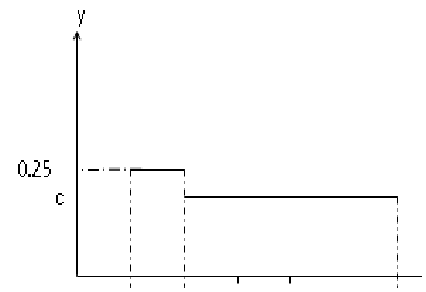
The Continuous Random Variable

General Probabilities without Integrals

Questions

1) X is a continuous variable having the density function shown below:

a. Find the value of c.
b. Construct the cumulative distribution function.
c. Calculate the following probabilities:
   i. $P(X=4)$
   ii. $P(X>1.5)$
   iii. $P(1.5<X<5)$
   iv. $P(5<X<10)$
d. Find the median of X.



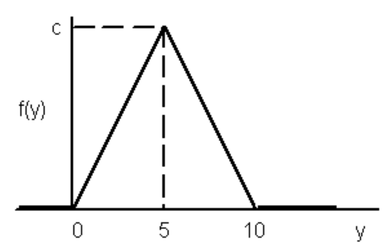
2) A continuous random variable X has the following density function:
$f(x)=cx$, where $0 \le x \le b$, 0 otherwise.

It is known that $P(0<X<1)=\frac{1}{4}$.

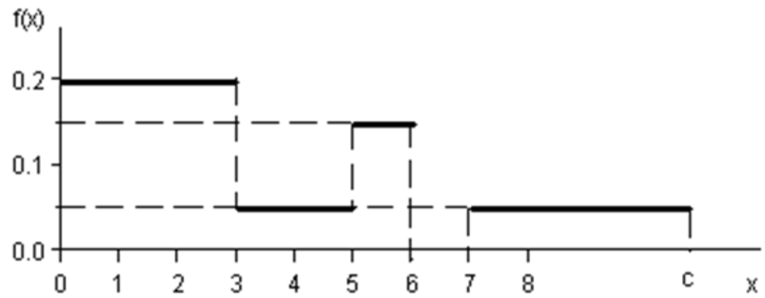
a. Write the density function of X.
b. Find the median of X.
c. What are the chances of X being less than 0.5?

3) The diagram below shows the density function of the random variable Y

a. Calculate c.
b. Write the cumulative distribution function of Y.
c. Calculate the probabilities: $P(Y>4)$, $P(7.5<X<15.5)$, $P(Y \le 3)$, $P(Y=7)$.
d. Calculate the bottom 10^{th} percentile $Y_{0.1}$, the bottom 25^{th} percentile $Y_{0.25}$, and the median of Y and the upper 10^{th} percentile $Y_{0.9}$.

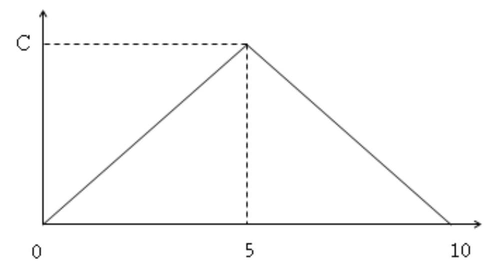


4) The diagram below shows the density function of the random variable X :



a. Calculate the value of c that define the diagram as a density function.
b. Write the cumulative distribution function of X .
c. Calculate the probabilities: $P(1<X\le 5)$, $P(X\ge -2)$, $P(X\ge 4)$.

5) Given the following density function
a. What is the value of c ?
b. Find a symmetric interval around the value 5 in which the probability equals 0.5.



6) The waiting time in minutes of a customer in line at the neighborhood supermarket has the following cumulative distribution function: $F(t)=1-e^{-0.2t}$.
a. What are the chances that the waiting time will be less than 15 minutes?
b. What is the probability of a customer waiting in line a total of less than 15 minutes if he has already waited in line for 10 minutes?
c. What is the time under which 90% of the customers have to wait?

Answer Key

1) a. $\frac{3}{16}$

b. $F(t)=P(x\le t)=\begin{cases}0 & x<1\\ 0.25(t-1) & 1\le x\le 2\\ 0.25+\frac{3}{16}(t-1) & 2<x<6\\ 1 & x>6\end{cases}$

c.(i) $\frac{5}{8}$ (ii) $\frac{7}{8}$ (iii) $\frac{11}{16}$ (iv) $\frac{3}{16}$ d. $3\frac{1}{3}$

2) a. $f(x)=\begin{cases}\frac{1}{2}x & 0<x\le 2\\ 0 & \text{other}\end{cases}$ b. 1.41 c. $\frac{1}{16}$

3) a. $\frac{1}{5}$

b. $F(t)=P(y\le t)=\begin{cases}0 & y<0\\ 0.02\text{t}^2 & 1\le y\le 5\\ 1\text{-}(t-10)^2\cdot 0.02 & 5<y\le 10\\ 1 & y>10\end{cases}$

c. $\text{P}(y=7)=0$; $\text{P}(y\le 3)=0.18$; $\text{P}(7.5<y<15.5)=0.125$ $\text{P}(y>4)=0.32$

d. $P(Y\le y_{0.1})=0.1\Rightarrow t=\sqrt{5}\,;P(Y\le y_{0.25})=0.25\Rightarrow t=\sqrt{12.5}$

$Y_{0.9}=7.76$; $\text{median}=5$

4) a. $c=10$

b. $F(t)=\begin{cases}t<0 & 0\\ 0\le t\le 3 & 0.2\text{t}\\ 3<t\le 5 & 0.6+0.05(t-3)\\ 5<t\le 6 & 0.7+0.15(t-5)\\ 6<t\le 7 & 0.85\\ 7<t\le 10 & 0.85+0.05(t-7)\\ t>10 & 1\end{cases}$

c. $P(X\ge 4)=0.35$; $P(X\ge -2)=1$; $P(1\le X\le 5)=0.5$

5) a. $c=0.2$ b. 5 ± 1.46

6) a. $P(x\ge 15)=0.0498$ b. $P(x<15|x>10)=0.6321$ c. $t=x_{0.9}=115.13$

Exponential Probability

Questions

1) The time until a malfunction occurs in the system has an exponential probability distribution with an expectation of 0.5 hours.

a. What is the probability that the next malfunction will occur in more than 0.5 hours?
b. What is the probability that the next malfunction will occur in less than an hour?
c. Find the median time for the occurrence of a malfunction in the system.

2) On a given highway the duration between accidents is distributed exponentially with and expectation time for 24 hours.

a. What is the standard deviation of the time until the next accident?
b. What is the probability that the next accident will occur in less than 24 hours?
c. What is the probability that the next accident will occur in less than two days?

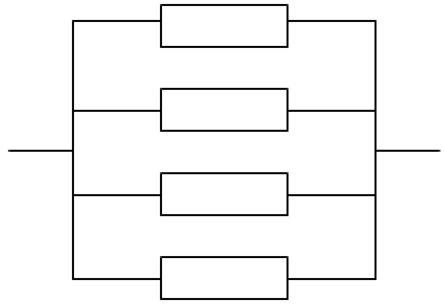
3) The time that students work continuously on a computer is exponentially distributed with an expectation of 30 minutes.

a. What are the chances that a student's work on the computer will last less than 15 minutes?
b. What are the chances that a student's work on the computer will last between 15 and 30 minutes?
c. If a student has already been working on the computer for more than 10 minutes, what is the probability that his overall work duration will exceed 30 minutes?
d. What is the time under which a student will complete 90% of his work?

4) An average of four patients per hour arrive at the emergency room in a Poisson flow.

a. Betty, the secretary, comes to the emergency room. What is the probability that the time she waits for the next patient is more than 20 minutes?
b. If Betty waited more than 15 minutes for the next patient, what is the probability that she will have to wait a total of more than 30 minutes?
c. What is the probability that more than 15 minutes will have passed between the first and second patients and less than 15 minutes between the second and third patients?

5) The following diagram illustrates an electrical system that has four identical electronic components that are operating simultaneously:



The system will properly when at least one component works properly. The lifespan of each component has an exponential probability distribution with an average of 100 hours.

a. What is the probability of the system operating properly for at least 100 hours?
b. The designers of the system are considering adding another component to the system. The cost of the extra component is $\$K$.
If the system works less than 100 hours, it causes damages totaling $\$A$.
c. What is the condition whereby adding another component is feasible?

Answer Key

1)	a. 0.368	b. 0.865	c. 0.347	
2)	a. $\sigma = 24$	b. 0.632	c. 0.135	
3)	a. 0.393	b. 0.239	c. 0.513	d. 69.08
4)	a. 0.264	b. 0.368	c. 0.232	
5)	a. 0.8403	b. $K < 0.0588A$		

Uniform Probability

Questions

1) The duration of a recess in school is distributed $U(13,16)$.

a. What are the expectation and standard deviation of the recess duration?
b. What is the probability that the recess lasts for over 15 minutes?
c. What is the probability that the duration of the recess will deviate from the expectation by less than a minute?

2) A train reaches the station every 10 minutes during the day. You arrive at a random time to the station.

a. Explain the probability distribution of the waiting time for the train.
b. If you have to wait longer than 5 minutes for the train, what is the probability y that you will have to wait for a total of less than eight minutes?
c. What is the expected number of days that will pass before you have to wait more than nine minutes for the train?

3) An ice cream machine automatically fills a cone with ice cream. The weight of the ice cream in the cone has a uniform probability distribution between 100 and 110 grams (the weight of the ice cream without the cone).

a. What is the probability that the weight of the ice cream in the cone will be more than 108 grams?
b. Assuming that the ice cream in the cone weighs less than 107 grams, what is the probability that it weighs more than 105 grams?
c. What is the top 10^{th} percentile of the weight of the ice cream in the cone?

Answer Key

1)	a. $E(X) = 14.5$, $\sigma(X) = 0.866$	b. $\frac{1}{3}$	c. $\frac{2}{3}$
2)	a. $X \sim U(0,10)$	b. 0.6	c. 10
3)	a. 0.2	b. $\frac{2}{7}$	c. 109

Normal Probability

Questions

1) The height of people in a given population has a normal probability distribution with an average of 170cm and a standard deviation of 10cm.

a. What is the proportion of people who are shorter than 182.4cm?
b. What is the proportion of people who are taller than 190cm?
c. What is the proportion of people who are exactly 173.6cm tall?
d. What is the proportion of people who are shorter than 170cm?
e. What is the proportion of people who are at most 170cm tall?

2) Assume that the time it takes for a certain medication to take effect has a normal probability distribution with an average of 30 minutes and a variance of nine minutes.

a. What is the proportion of cases where the medication takes longer than an hour to work?
b. What is the proportion of cases where the medication takes between 35 and 37 minutes to work?
c. What are the chances that the medication will help after exactly 36 minutes?
d. What is the proportion of cases where the time taken by the medication to work deviates from 30 minutes by less than three minutes?

3) The weight of people in a given population has a normal probability distribution with an average of 60kg and a standard deviation of 8kg.

a. What is the proportion of people who weigh less than 55kg?
b. What is the proportion of people in the population who weigh less than 50kg?
c. What is the relative frequency of the people in the population who weigh between 60 and 70kg?
d. What is the proportion of the population whose weight deviates from the average by no more than 4kg?
e. What are the chances of a randomly selected person weighing less than 140kg?

4) The weight of babies at birth has a normal probability distribution with an average of 3300 grams and a standard deviation of 400 grams.

a. Find the upper 10^{th} percentile.
b. Find the 95^{th} percentile.
c. Find the bottom 10^{th} percentile.

5) Marks in an intelligence test have a normal probability distribution with an average of 100 and a variance of 225.

a. What is the upper 10^{th} percentile of the marks on the intelligence test?
b. What is the bottom 10^{th} percentile of the probability distribution?
c. 20% of those taking the test receive marks higher than what number?
d. What is the 20^{th} percentile?
e. 5% of those taking the test receive marks lower than what number?

6) The volume of a bottled beverage has a normal distribution with a standard deviation of 20ml. Assume that 33% of the bottles have a volume of over 508.8ml.

a. What is the average volume of a bottled beverage?
b. 5% of the bottles that are produced with the largest volume are sent for testing. Starting from what volume are bottles sent for testing?
c. 1% of the bottles with the lowest volume are donated to charity. What is the maximum volume of the bottles donated to charity?

7) The lifespan of a device has a normal probability distribution. It is known that half of the devices last less than 500 hours, and that 67% of the devices last less than 544 hours.

a. What is the average lifespan of a device?
b. What is the standard deviation of the lifespan of a device?
c. What are the chances that a randomly selected device will last less than 460 hours?
d. What is the upper 1 percentile of a device's lifespan?
e. 1% of the devices with shortest lifespan are sent to the laboratory for a thorough check. What is the maximum lifespan of a device sent to the laboratory?

8) The following are three normal probability distributions of three different groups sketched on system of coordinate axes.

a. Which probability distribution has the highest average?
b. In which of the following measures are distributions 1 and 2 the same?
   i. In their upper 10^{th} percentile.
   ii. In their average.
   iii. In their variance.
c. Which distribution has the smallest standard deviation?
   i. 1
   ii. 2
   iii. 3
   iv. No option.

9) The time it takes a person to get to work has a normal probability distribution with an average of 40 minutes and a standard deviation of five minutes.

a. What is the probability that it takes less than 45 minutes for a person to get to work?
b. A person leaves home to go to work at $8:10$. He has to get there by $9:00$. What are the chances of him being late?
c. If it is known that it takes a person longer than 45 minutes to get to work, what is the probability that the total time it took him is less than 50 minutes?
d. What are the chances it will take a person at least 45 minutes to get to work at least once during a five-day work week?

10) The monthly household spending in the city of Tarera has a normal probability with an average of $\$2,000$ and a standard deviation of $\$300$. Five households are randomly selected. The probability that at least one of them spends more than T dollars per month is 0.98976.

a. What is the value of T ?
b. What are the chances that a household in the town spends at least one standard deviation more than T ?
c. It is learned that a mistake was made in the data, and $\$100$ must be added to the monthly spending of all the households in the city.
Given this correction, what is the probability that a household's monthly spending is less than $\$1,800$?

11) The length of a random song that is broadcasted on the radio has a normal probability distribution with an expectation of 3.5 minutes and a standard deviation of 30 seconds.

a. What is the probability that the length of a random song played on the radio is between 2.5 and 3 minutes?
b. What is the inter-quartile range of the length of a song broadcast on the radio?
c. 200 songs are played on the radio on a given day.
How many songs shorter than 3.5 minutes can we expect to be played?
d. Eight songs are broadcasted during a given hour. What is the probability that exactly a quarter of them were longer than four minutes, and the rest were no longer?

Answer Key

1)	a. 89.25%	b. 2.28%	c. 0	d. 50%	e. 50%
2)	a. 0%	b. 3.76%	c. 0	d. 68.26%	
3)	a. 26.43%	b. 89.44%	c. 39.44%	d. 38.3%	e. $\cong 1$
4)	a. 3812.8	b. 3958	c. 2787.2		
5)	a. 119.23	b. 80.77	c. 112.6	d. 87.4	e. 75.325
6)	a. 500	b. 532.9	c. 453.48		
7)	a. 500	b. $\sigma = 100$	c. 0.3446	d. 732.6	e. 267.4
8)	a. 3	b. In their average	c. 1		
9)	a. 0.1587	b. 0.0228	c. 0.1359	d. 0.3975	
10)	a. $T = 1925$	b. 0.2266	c. 0.1587		
11)	a. 0.1359	b. 0.675	c. $\mathrm{E}(y) = 100$	d. 0.25	

Transformation of a Continuous Random Variable

Questions

1) Let W be a random variable with an exponential distribution whose expectation is 1.
We define a new variable: $Y = e^{x}$.

a. Find the cumulative distribution function of Y.
b. Identify Y as a special probability distribution and find the parameters.

2) Assume that $X \sim \mathrm{U}(0,1)$.

A new variable R is defined where $R = X^2$.
Find the density function of the new variable R.

3) Assume that $X \sim \exp(\lambda)$ and $Y = \ln(X)$.

Prove that the density function of Y is given by the following formula: $f(Y) = \lambda \cdot e^{-\lambda \cdot e^{Y}+1}$

4) Assume that $X \sim \exp(\lambda = 1)$, and let $Y = 1 - 2 \cdot e^{-X}$.

a. Find the cumulative distribution function of Y.
b. Identify the probability distribution of Y.

5) The length of the side of a die has uniform probability between 1 and 2.
Find the density function of the die's volume.

6) Assume the following cumulative distribution function: $F_X(t) = \theta^{t} - 1$ for $0 \le t < 1$.

a. Find the value of the parameter θ.
b. Find the density function of X.
c. Let $Y = 2^{X} - 1$.
Find the density function of Y, and identify the probability.

Answer Key

1) a. $F(Y)=y$ b. $Y\sim U(a=0,b=1)$

2) $f(R)=\dfrac{1}{2\sqrt{r}}$ where $0<r<1$

3) $\lambda e^{-\lambda e^{y}+1}$

4) a. $F(y)=\begin{cases} y<-1 & 0 \\ -1\le y\le 1 & \dfrac{y+1}{2} \\ y>1 & 1 \end{cases}$ b. $Y\sim \mathrm{U}(a=-1,\mathrm{b}=1)$

5) $f(Y)=\dfrac{1}{3\sqrt[3]{y^2}}$ where $1\le y\le 8$

6) a. $\theta=2$ b. $f(x)=2^x\ln 2$ where $0\le x\le 1$ c. $Y\sim U(a=0,\mathrm{b}=1)$