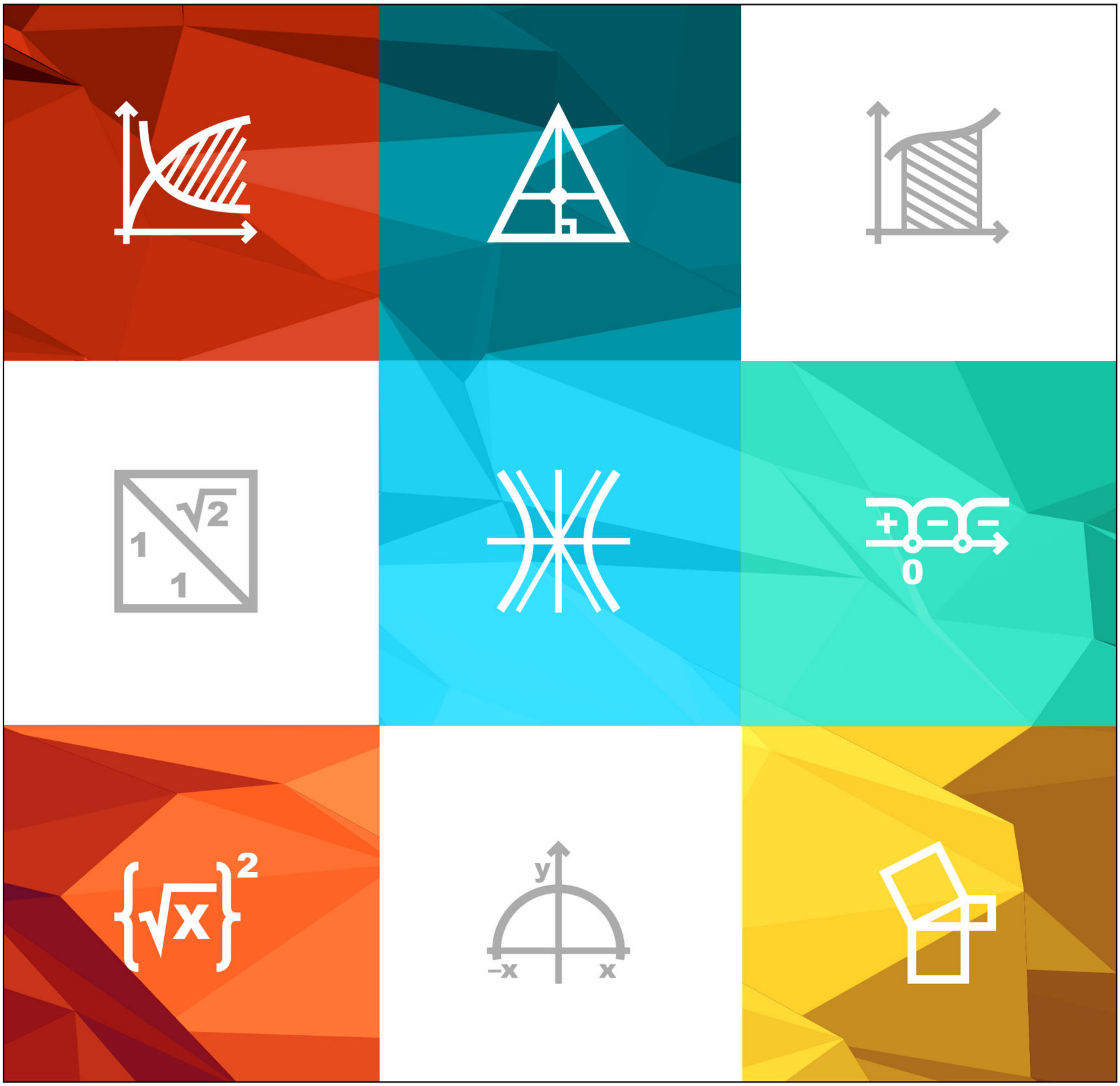
proprep™

Workbook



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Rolle`s Theorem and the Mean Value Theorem

The Mean Value Theorem

Questions:

1) Prove that $\frac{b-a}{b} < \ln\left(\frac{b}{a}\right) < \frac{b-a}{a} \qquad (0<a<b)$.

2) Prove that $\frac{b-a}{\cos^2 a} < \tan b - \tan a < \frac{b-a}{\cos^2 b} \qquad \left(0<a<b<\frac{\pi}{2}\right)$.

3) Prove that $(a-b)e^{-a} < e^{-b} - e^{-a} < \quad (a-b)e^{-b} \qquad (a<b)$.

4) Prove that $\frac{b-a}{1+b^2} < \arctan b - \arctan a < \frac{b-1}{1+a^2} \qquad (0>a>b)$.

5) Prove that $\frac{b-a}{\sqrt{1-a^2}} < \arcsin b - \arcsin a < \frac{b-a}{\sqrt{1-b^2}} \qquad (0<a<b<1)$.

6) Prove that $\frac{b-a}{\sqrt{1+b^2}} < \frac{\text{arcsinh}(b) - \text{arcsinh}(a)}{b-a} < \frac{b-a}{\sqrt{1+a^2}} \qquad (0<a<b)$.

7) Prove that $\frac{b-a}{1-a^2} < \text{arctanh}(b) - \text{arctanh}(a) < \frac{b-a}{1-b^2} \qquad (0<a<b<1)$.

8) Prove that $\sqrt[n]{b} \cdot \frac{b-a}{n \cdot b} < \sqrt[n]{b} - \sqrt[n]{a} < \sqrt[n]{a} \cdot \frac{b-a}{n \cdot a} \qquad (0<a<b)$.

9) Prove that $\frac{2b(b-a)}{b^2+1} < \ln\left(\frac{b^2+1}{a^2+1}\right) < \frac{2a(b-a)}{a^2+1} \qquad (0<a<b)$.

10) Prove that $x < \tan x < \frac{x}{\cos^2 x} \qquad \left(0<x<\frac{\pi}{2}\right)$.

11) Prove that $\frac{x}{1+x^2} < \arctan x < x \qquad (x>0)$.

12) Prove that $\frac{x}{\sqrt{1+x^2}} < \text{arcsinh}(x) < x \qquad (x>0)$.

13) Prove that $x < \text{arctanh}(x) < \frac{x}{1-x^2} \qquad (0<x<1)$.

14) Prove that $1+x < e^x < 1+xe^x \qquad (x>0)$.

15) Prove that $\sin x \le x \quad (x > 0)$.

16) Prove that $\tan x < 4x \quad \left(0 < x < \frac{\pi}{3}\right)$.

17) Prove that $\arctan x > \ln(1+x) \quad (0 < x < 1)$.

18) Prove that $|\sin x_2 - \sin x_1| \le |x_2 - x_1|$.

19) Prove that $|\cos x_2 - \cos x_1| \le |x_2 - x_1|$.

20) Prove that $|\arctan y - \arctan x| < |y - x|$.

21) Prove that $|\tan y - \tan x| \le 8|\sin y - \sin x|, \quad x, y \in \left[0, \frac{\pi}{3}\right]$.

22) Prove that $\frac{1}{3} < \ln\left(\frac{3}{2}\right) < \frac{1}{2}$.

23) Prove that $\frac{1}{2\sqrt{2}} + 1 < \sqrt{2} < 1.5$.

24) Prove that $\frac{3}{25} + \frac{\pi}{4} < \arctan\left(\frac{4}{3}\right) < \frac{1}{6} + \frac{\pi}{4}$.

25) Prove that $\frac{\sqrt{3}}{15} + \frac{\pi}{6} < \arcsin(0.6) < \frac{1}{8} + \frac{\pi}{6}$.

26) Let $f(x)$ be a function which is differentiable for all x and satisfies $|f'(x)| \le 5$.
It is known that $f(1) = 3$, $f(4) = 18$. Prove that $f(2) = 8$.

27) Let $f(x)$ be a function which is differentiable for all x and satisfies $|f'(x)| \le 7$.
It is known that $f(1) = 3$, $f(4) = 18$. Prove that $4 \le f(2) \le 10$.

Answer Key:

Refer to the videos.

Rolle's Theorem

Example Questions:

1) Check if the given function f satisfies the conditions of Rolle's Theorem on the given interval. If so, find all the values of c as in the conclusion of Rolle's Theorem.

a. $f(x)=x^3-3x^2+2x$ on $[0,2]$.

b. $f(x)=\dfrac{x^2-1}{x-2}$ on $[-1,1]$.

2) Given: $f(x)=\dfrac{1}{(x-3)^2}$. Show that $f(1)=f(5)$, but that there is no value c in $1<c<5$ such that $f'(c)=0$. Does this contradict Rolle's Theorem ?

3) Prove that the equation $x^2+x^3+5x=1$ has exactly one solution.
Hint: show at least one solution and at most one solution.

Answer Key:

1) a. $f(0)=0,\ f(2)=0$; Yes, it is continuous and differentiable

$$c=\frac{3\pm\sqrt{3}}{3}\cong 1.6 \ and\ -0.4$$

b. $f(-1)=0,\ f(1)=0$; Yes, it is continuous and differentiable

$$c=2-\sqrt{3}$$

2-3) Refer to the video.