Workbook

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Early Transcendentals – 14th Edition

Integrals and Vector Fields

Line Integrals of Scalar Functions

Questions

1) Compute the following:

a. $\int_C (1-x^2)\,ds$, where $C:\ x=\cos t,\ \ y=\sin t,\ \ 0\le t\le 2\pi$.

b. $\int_C x\,ds$, where $C:\ x=t-\sin t,\ \ y=1-\cos t,\ \ 0\le t\le \pi$.

c. $\int_C (x+y)\,ds$, where C is the line segment joining $\mathrm{O}(0,0)$ with $\mathrm{A}(1,2)$.

d. $\int_C (x+y^2)\,ds$, where C is the perimeter of $\Delta\mathrm{OAB}:\mathrm{O}(0,0)$, $\mathrm{A}(0,1)$, $\mathrm{B}(1,0)$

2) Compute the following:

a. $\int_C (x^2+y^2+z^2)\,ds$, where $C: x=\cos t,\ y=\sin t,\ z=t\ \ 0\le t\le \pi$.

b. $\int_C (x^3+3z)\,ds$, where $C:\ x=t,\ \ y=\frac{1}{\sqrt{2}}t^2,\ \ z=\frac{1}{3}t^3\ \ \ 0\le t\le 3$.

3) Compute the length of the curve: $x^{\frac{2}{3}}+y^{\frac{2}{3}}=1$.

4) A coil made of thin wire is expressed by $x=\cos t,\ y=\sin t,\ z=2t\ \ (0\le t\le \pi)$.

Compute the **mass** of the coil if the density function is $\delta(x,y,z)=kz\ \ (k>0)$.

5) Compute the following:

a. $\int_C 2xy\,dx+(x^2+y^2)\,dy\ \ ;\ \ C:\ x=\cos t,\ \ y=\sin t\ \ 0\le t\le \frac{\pi}{2}$

b. $\int_C (2x+y)\,dx+(x^2-y)\,dy\ \ ;\ \ C:\ \ x=t,\ \ y=t^2\ \ 0\le t\le 1$

6) Compute $\int_C ydx + x^2dy$, where C is the path from point $(0,0)$ to point $(2,4)$, given by the equation:

a. $y = 2x$

b. $y = x^2$

7) Compute $\int_{(1,1)}^{(4,2)} (x+y)dx + (y-x)dy$ along each of the following curves:

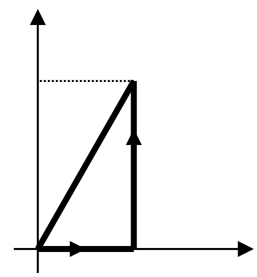
a. The parabola $y^2 = x$.

b. A line segment.

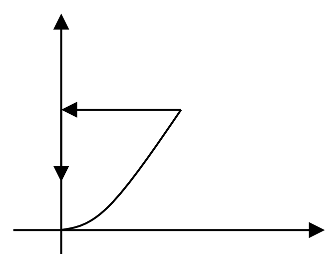
c. The line segments from $(1,1)$ to $(1,2)$ and then to $(4,2)$.

d. The curve: $x = 2t^2 + t + 1, \quad y = t^2 + 1 \quad 0 \le t \le 1$.

8) Compute $\int_C x^2ydx + xdy$, where the path C is as in the figure:



9) Compute $\int_C (x - y^2)dx + dy$, where C is as in the figure:



10) If $\mathbf{F}(x,y,z) = (3x^2 - 6yz)\mathbf{i} + (2y + 3xz)\mathbf{j} + (1 - 4xyz^2)\mathbf{k}$,

compute the line integral $\int_C F \cdot dr$ from $(0,0,0)$ to $(1,1,1)$ along the paths:

a. $x = t,\ y = t^2,\ z = t^3$

b. The lines from $(0,0,0)$ to $(0,0,1)$, then to $(0,1,1)$ and then to $(1,1,1)$.

c. The line from $(0,0,0)$ to $(1,1,1)$.

11) Compute the line integral $\int_C F \cdot dr$, where:

a. $F(x,y)=\langle x^2y^3,-y\sqrt{x}\rangle, \quad C: r(t)=\langle t^2,-t^3\rangle, \quad 0\le t\le 1$

b. $F(x,y,z)=\langle \sin x,\cos y, xz\rangle$, $C: r(t)=\langle t^3,-t^2,t\rangle, \quad 0\le t\le 1$

12) Answer the following:

a. Compute the work done by the force field $\mathbf{F}(x,y)=x^3y\mathbf{i}+(x-y)\mathbf{j}$ on a particle which moves along the parabola $y=x^2$ from $(-2,4)$ to $(1,1)$.

b. How would your answer change, if the particle moved from $(1,1)$ to $(-2,4)$?

13) Compute the work done by the force field $\mathbf{F}(x,y,z)=yz\mathbf{i}+xz\mathbf{j}+xy\mathbf{k}$ on a particle which moves along the path $\mathbf{r}(t)=t\mathbf{i}+t^2\mathbf{j}+t^3\mathbf{k}$ $(0\le t\le 1)$.

Remark on Notation

A line integral of type II has different notation in the technical literature:

$$\int_C \mathbf{F}\cdot dr=\int_C (f,g,h)\cdot(dx,dy,dz)=\int_C fdx+gdy+hdz$$

$$\int_C \mathbf{A}\cdot dr=\int_C (A_1,A_2,A_3)\cdot(dx,dy,dz)=\int_C A_1dx+A_2dy+A_3dz$$

Answer Key

1) a. π b. $\frac{16}{3}$ c. $\frac{3\sqrt{5}}{2}$ d. $\frac{3\sqrt{5}}{2}$

2) a. $2\sqrt{2}\pi\left(1+\frac{4\pi^2}{3}\right)$ b. $\frac{567}{2}$

3) 6

4) $\sqrt{5}k\pi^2$

5) a. $\frac{1}{3}$ b. $\frac{4}{3}$

6) a. $\frac{28}{3}$ b. $\frac{32}{3}$

7) a. $\frac{34}{3}$ b. 11 c. 14 d. $\frac{32}{3}$

8) $\frac{1}{2}$

9) $\frac{4}{5}$

10) a. 2 b. -3 c. $\frac{6}{5}$

11) a. $-\frac{59}{105}$ b. $\frac{6}{5}-\sin 1-\cos 1$

12) a. 3 b. -3

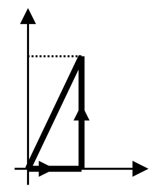
13) 1

Green's Theorem in the Plane

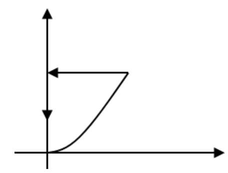
Questions

In each of the exercises **1-3** verify Green's Theorem: $\oint_C fdx + gdy = \iint_R \left(g_x - f_y\right)dA$.

1) $\oint_C x^2 ydx + xdy$; the path C is as in the illustration:



2) $\oint_C (x - y^2)dx + dy$; the path C is as in the illustration:



3) $\oint_C \left(x^2 - xy^3\right)dx + \left(y^2 - 2xy\right)dy$; C traces out, anticlockwise, the square with vertices: $(0,0),(2,0),(2,2),(0,2)$.

4) Compute the work done by a force field $\mathbf{F}(x,y) = \left(e^x - y^3\right)\mathbf{i} + \left(\cos y + x^3\right)\mathbf{j}$, on a particle which moves anticlockwise on the unit circle $x^2 + y^2 = 1$ and completes one revolution.

5) Compute the integral $\int_C \left(e^y - \tan\frac{x}{2}\right)dx + \left(xe^y + y\cos y^2\right)dy$, where C is the clockwise union of the parts of the curves $y = 8 - x^2$, $y = x^2$ between the y-axis and their intersection in the first quadrant.

6) Compute the integral $\int_C -2e^{2x-y}\cos ydx + \left(e^{2x-y}\left(\sin y + \cos y\right) + 2xy\right)dy$ where C is the semi-ellipse $\left\{x^2 + 4y^2 = 4,\ y \geq 0\right\}$ from the point $(2,0)$ to the point $(-2,0)$.

7) Answer the following:

a. Prove that the area bounded by a simple closed curve C is given by $\frac{1}{2}\oint_C xdy - ydx$.

b. Compute the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, using the formula $A = \frac{1}{2}\oint_C xdy - ydx$.

Answer Key

1) The common value is 0.5.

2) The common value is 0.8.

3) The common value is 8.

4) 1.5π

5) $0.5\sin 64$

6) $\frac{8}{3}+2\left(e^4-\frac{1}{e^4}\right)$

7) a. Solution in the recording. b. πab

Surface Integrals

Questions

1) Compute $\iint_S x^2 yz\,dS$, where S is the part of the plane $z = 1 + 2x + 3y$ above the rectangle $R = [0,3] \times [0,2]$.

2) Compute $\iint_S x\,dS$, where S is the surface $0 \le x \le 2$, $0 \le z \le 2$, $y = x^2 + 4z$.

3) Compute $\iint_S yz\,dS$, where S is the part of the plane $z = y + 3$ inside the cylinder $x^2 + y^2 = 1$.

4) Compute $\iint_S \left(x^2 z + y^{2z}\right) dS$, where S is the hemisphere $x^2 + y^2 + z^2 = 4, \quad z \ge 0$.

5) Compute $\iint_S xyz\,dS$, where S is the part of the cone $\mathbf{r}(u,v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} + 3u\mathbf{k}$ satisfying $1 \le u \le 2$, $0 \le v \le \frac{\pi}{2}$.

6) Compute the surface area of a sphere with radius R, $\iint_S \left(x^2 z + y^2 z\right) dS$, where S is the hemisphere $x^2 + y^2 + z^2 = 4, \quad z \ge 0$.

7) The thin sheet S is the part of the paraboloid $z = x^2 + y^2$ below the plane $z = 1$ and it has a constant density $\delta(x,y,z) = \delta_0$.
Compute the mass of the sheet.

In each of the exercises **8-12** compute $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$, where $\mathbf{n}$ is the outward unit normal to S.

8) $\mathbf{F}=\left(\frac{x^2y}{1+y^2}+6yz^2\right)\mathbf{i}+\arctan y\mathbf{j}-\frac{2xz(1+y)+1+y^2}{1+y^2}\mathbf{k}$;
S is the open surface $z=4-x^2-y^2$, $z\geq 0$.

9) $\mathbf{F}=x\mathbf{i}-2y\mathbf{j}+3z\mathbf{k}$; S is the sphere $x^2+y^2+z^2=1$.

10) $\mathbf{F}=(2xy+z)\mathbf{i}+y^2\mathbf{j}-(x+3y)\mathbf{k}$; S is the surface of the pyramid determined by the planes: $2x+2y+z=6,\ x=0,\ y=0,\ z=0$.

11) $\mathbf{F}=5\mathbf{i}+2\mathbf{j}+3\mathbf{k}$; S is the part of the paraboloid $z=4-x^2-y^2$, where $z\geq 0$.

12) $\mathbf{F}=0\mathbf{i}-2z\mathbf{j}+(-3y-1)\mathbf{k}$; S is the hemisphere centered at the origin, with radius 4, and above the xy-plane.

Answer Key

1) $171\sqrt{14}$	**2)** $\frac{33\sqrt{33}-17\sqrt{17}}{6}$	**3)** $\frac{\pi\sqrt{2}}{4}$
4) 16π	**5)** $\frac{93}{\sqrt{10}}$	**6)** $4\pi R^2$
7) $\frac{\pi\delta_0}{6}(5\sqrt{5}-1)$	**8)** -4π	**9)** $\frac{8\pi}{3}$
10) $37\frac{1}{2}$	**11)** 12π	**12)** -16π

Stokes' Theorem

Questions

In each of the exercises **1-2** verify Stokes' Theorem: $\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\text{curl}\mathbf{F})\cdot \mathbf{n}dS$.

(See remark on notation below)

1) $\mathbf{F} = 2z\mathbf{i} + 3x\mathbf{j} + 5y\mathbf{k}$; S is the part of the paraboloid $z = 4 - x^2 - y^2$ where $z \geq 0$.

2) $\mathbf{F} = (x^2 + y - 4)\mathbf{i} + (-3xy)\mathbf{j} + (2xz + z^2)\mathbf{k}$; S is the half of the sphere centered at the origin with radius 4, that lies above the xy plane.

3) Compute the integral $\oint_C x^2dx + 4xy^3dy + y^2xdz$, where C is the rectangle-shaped curve from $(0,0,0)$ to $(0,3,3)$ to $(1,3,3)$ to $(1,0,0)$.

4) Compute the integral $\oint_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = (x + y^2)\mathbf{i} + (y + z^2)\mathbf{j} + (z + x^2)\mathbf{k}$ and C is the perimeter of the triangle with vertices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ and in the anticlockwise direction (as seen above from the positive z-axis).

5) Compute $\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}dS$, where $\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ and S is the part of the sphere $x^2 + y^2 + z^2 = 4$ inside the cylinder $x^2 + y^2 = 1$ and above the xy plane.

6) Compute $\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}dS$, where $\mathbf{F} = (x - z)\mathbf{i} + (x^3 + yz)\mathbf{j} - 3xy^2\mathbf{k}$ and S is the part of the cone $z = 2 - \sqrt{x^2 + y^2}$ lying above the xy-plane.

Remark on Notation

Stokes' Theorem states that if $\mathbf{F}(x,y,z)=f(x,y,z)\mathbf{i}+g(x,y,z)\mathbf{j}+h(x,y,z)\mathbf{k}$ is a vector-field, then we have the equality: $\oint_C \mathbf{F}\cdot d\mathbf{r}=\iint_S (\mathrm{curl}\mathbf{F})\cdot\mathbf{n}dS$.

There are various other formulations of the Divergence Theorem, such as:

$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\iint_S (\mathrm{curl}\mathbf{F})\cdot\mathbf{n}dS$$
$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\iint_S (\mathrm{Rot}\mathbf{F})\cdot\mathbf{n}dS$$
$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}dS$$
$$\oint_C fdx+gdy+hdz=\iint_S \left((h_y-g_z)\mathbf{i}+(f_z-h_x)\mathbf{j}+(g_x-f_y)\mathbf{k}\right)\cdot\mathbf{n}dS$$

Answer Key

1) The common value is 12π .
2) The common value is -16π .
3) -90
4) -1
5) 0
6) 12π

The Divergence Theorem and a Unified Theory

Questions

In each of the exercises **1-3** verify the Divergence Theorem $\left(\iiint_R div\mathbf{F}dV = \iint_S \mathbf{F}\cdot\mathbf{n}dS \right)$.

$\mathbf{n}$ is the outward unit normal to S (See remark on notation below).

1) Where $\mathbf{F} = (2x - z)\mathbf{i} + x^2y\mathbf{j} - xz^2\mathbf{k}$ and S is the surface of the cube R determined by the planes: $x=0,\ x=1,\ y=0,\ y=1,\ z=0,\ z=1$.

2) Where $\mathbf{F} = x\mathbf{i} - 2y\mathbf{j} + 3z\mathbf{k}$ and S is the surface of the unit ball. $x^2+y^2+z^2 \le 1$ given by R

3) Where $\mathbf{F} = (2xy + z)\mathbf{i} + y^2\mathbf{j} - (x+3y)\mathbf{k}$ and S is the surface of the pyramid R determined by the planes: $2x+2y+z=6,\ x=0,\ y=0,\ z=0$.

4) Let S be the surface of the body bounded by the cylinder $x^2+y^2=9$ and the planes $z=0$ and $z=2$. Compute the flux of the vector field $\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + z^2\mathbf{k}$ through S. I.e. compute $\iint_S \mathbf{F}\cdot\mathbf{n}dS$ where $\mathbf{n}$ is the outward unit normal to S.

5) Compute $\iint_S \mathbf{F}\cdot\mathbf{n}dS$ where $\mathbf{n}$ is the outward unit normal to S, $\mathbf{F} = (z^2 - x)\mathbf{i} - xy\mathbf{j} + 3z\mathbf{k}$, and S is the surface of the body bounded by: $x=0,\ x=3,\ z=4-y^2,\ z=0$.

6) Compute $\iint_S \mathbf{F}\cdot\mathbf{n}dS$ where $\mathbf{n}$ is the outward unit normal to S

$\mathbf{F} = xz^2\mathbf{i} + (x^2y - z^3)xy\mathbf{j} + (2xy + y^2z)\mathbf{k}$, and S is the surface of the body bounded by:

$z = \sqrt{a^2 - x^2 - y^2},\ z=0$.

7) Let S be the open surface $0 \le y \le 4$, $x^2+z^2=16$ (a cylinder without the bases). Compute the **flux** of the vector field $\mathbf{F} = z^2\mathbf{i} + 5y\mathbf{j} + x^5\mathbf{k}$ through S. I.e. compute $\iint_S \mathbf{F}\cdot\mathbf{n}dS$ where $\mathbf{n}$ is the outward unit normal to S.

8) Compute $\iint\limits_S \mathbf{F}\cdot\mathbf{n}dS$ where $\mathbf{n}$ is the outward unit normal to S,

$\mathbf{F}=\left(\dfrac{x^2y}{1+y^2}+6yz^2\right)\mathbf{i}+2x\arctan y\mathbf{j}-\dfrac{2xz(1+y)+1+y^2}{1+y^2}\mathbf{k}$, and S is the open surface $z=4-x^2-y^2,\quad z\ge 0$.

Remark on Notation:

The Divergence Theorem states that given a vector field $\mathbf{F}=f(x,y,z)\mathbf{i}+g(x,y,z)\mathbf{j}+h(x,y,z)\mathbf{k}$, the equality $\iiint\limits_R div\mathbf{F}dV=\iint\limits_S \mathbf{F}\cdot\mathbf{n}dS$ holds.

There are various other formulations of the Divergence Theorem, such as:

$$\iiint\limits_R \nabla\cdot\mathbf{F}dV=\iint\limits_S \mathbf{F}\cdot\mathbf{n}dS$$
$$\iiint\limits_R (f_x+g_y+h_z)dV=\iint\limits_S \mathbf{F}\cdot\mathbf{n}dS$$
$$\iiint\limits_R (f_x+g_y+h_z)dV=\iint\limits_S fdydz+gdzdx+hdxdy$$

Answer Key

1) The common value is $\frac{11}{6}$.

2) The common value is $\frac{8}{3}\pi$.

3) The common value is 27.

4) 279π

5) 16

6) $\frac{2\pi a^5}{5}$

7) 0

8) -4π