

Workbook

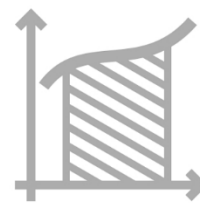


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Determinants

Determinants

Questions:

Evaluate the following determinants:

1) $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

2) $\begin{vmatrix} 5 & 2 \\ -7 & 3 \end{vmatrix}$

3) $\begin{vmatrix} 4 & -1.5 \\ 2 & -1 \end{vmatrix}$

4) $\begin{vmatrix} 1 & 0 & 2 \\ 4 & 1 & 8 \\ 2 & 0 & 3 \end{vmatrix}$

5) $\begin{vmatrix} 2 & 1 & 1 \\ 0 & 2 & -1 \\ 1 & 0 & 0 \end{vmatrix}$

6) $\begin{vmatrix} 2 & 1 & 1 \\ 3 & -2 & 5 \\ 0 & 2 & 0 \end{vmatrix}$

7) $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 4 \end{vmatrix}$

8) $\begin{vmatrix} 1 & 0 & 2 & 5 \\ -2 & 0 & -6 & 0 \\ 5 & 3 & -7 & 4 \\ 2 & 0 & 5 & 44 \end{vmatrix}$

9) $\begin{vmatrix} 4 & 0 & 0 & 5 \\ 1 & 7 & 2 & 4 \\ 4 & 0 & 0 & 0 \\ 1 & 4 & -1 & 1 \end{vmatrix}$

10) $\begin{vmatrix} 1 & 9 & 8 & 3 & 4 \\ 3 & 0 & -5 & 0 & 2 \\ 2 & -4 & 1 & 0 & 3 \\ 4 & 1 & 7 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \end{vmatrix}$

11) $\begin{vmatrix} 4 & 0 & 7 & 5 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ -7 & 2 & 1 & 5 & 9 \\ 3 & 0 & 4 & 2 & -1 \\ -5 & 0 & -8 & -3 & 2 \end{vmatrix}$

Answer Key:

1) $ad - bc$

2) 29

3) -1

4) -1

5) -3

6) -14

7) 24

8) 234

9) -300

10) 9

11) 6

Rules of Determinants

Questions:

1) Evaluate the following 4×4 determinants [3 of them] by using row operations:

a.
$$\begin{vmatrix} 1 & 3 & 0 & 2 \\ -2 & -5 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ 2 & 2 & 2 & -1 \end{vmatrix}$$

b.
$$\begin{vmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 5 & 4 & -3 \\ -1 & -2 & -1 & -1 \end{vmatrix}$$

c.
$$\begin{vmatrix} 1 & -1 & -3 & 0 \\ 1 & 0 & 2 & 4 \\ -1 & 2 & 8 & 5 \\ 3 & -1 & -2 & 3 \end{vmatrix}$$

Evaluate the following 5×5 determinants [3 of them] by using row operations:

d.
$$\begin{vmatrix} 1 & 3 & 0 & 2 \\ -2 & -5 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ 2 & 2 & 2 & -1 \end{vmatrix}$$

e.
$$\begin{vmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 5 & 4 & -3 \\ -1 & -2 & -1 & -1 \end{vmatrix}$$

f.
$$\begin{vmatrix} 1 & -1 & -3 & 0 \\ 1 & 0 & 2 & 4 \\ -1 & 2 & 8 & 5 \\ 3 & -1 & -2 & 3 \end{vmatrix}$$

Evaluate the following determinants:

2)
$$\begin{vmatrix} 2 & 5 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 9 \\ 6 & 15 & -7 & -2 \end{vmatrix}$$

3)
$$\begin{vmatrix} -1 & 2 & 3 & 0 \\ 3 & 4 & 3 & 0 \\ 5 & 4 & 6 & 6 \\ 3 & 4 & 7 & 3 \end{vmatrix}$$

4)
$$\begin{vmatrix} 2 & 5 & 4 & 1 \\ 6 & 12 & 10 & 3 \\ 6 & -2 & -4 & 0 \\ -6 & 7 & 7 & 0 \end{vmatrix}$$

5) Prove (without computing) that the following three determinants are all zero:

a.
$$\begin{vmatrix} 1 & 0 & 2 \\ 7 & 0 & 12 \\ 3 & 0 & 2 \end{vmatrix}$$

b.
$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 9 \end{vmatrix}$$

c.
$$\begin{vmatrix} 12 & 15 & 18 \\ 13 & 16 & 19 \\ 14 & 17 & 20 \end{vmatrix}$$

6) Prove (without computing) that the following three determinants are all zero:

a.
$$\begin{vmatrix} y+z & z+x & y+x \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

b.
$$\begin{vmatrix} a & a+x & a+y \\ b & b+x & b+y \\ c & c+x & c+y \end{vmatrix}$$

c.
$$\begin{vmatrix} \sin^2 x & \cos^2 x & 1 \\ \sin^2 y & \cos^2 y & 1 \\ \sin^2 z & \cos^2 z & 1 \end{vmatrix}$$

d.
$$\begin{vmatrix} 3 & -1 & 4 & 5 & 0 & 1 & -12 \\ -14 & 4 & 1 & -4 & 1 & 8 & 4 \\ 3 & 5 & -2 & 0 & -4 & 1 & -3 \\ -4 & 2 & 1 & 1 & 0 & 6 & -6 \\ -21 & 2 & 3 & 4 & 5 & 6 & 1 \\ 2 & -5 & 7 & -4 & 2.5 & -1 & -1.5 \\ -11 & 2 & -6 & 9 & -1 & 3 & 4 \end{vmatrix}$$

Given that:
$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 4$$
, evaluate the following determinants:

7)
$$\begin{vmatrix} a & g+d & 2d \\ b & h+e & 2e \\ c & i+f & 2f \end{vmatrix}$$

8)
$$\begin{vmatrix} 2a-3d & 2d & g+4a \\ 2b-3e & 2e & h+4b \\ 2c-3f & 2f & i+4c \end{vmatrix}$$

9)
$$\begin{vmatrix} 0 & g+3d & 3a & a+3d \\ 0 & h+3e & 3b & b+3e \\ 0 & i+3f & 3c & c+3f \\ 1 & 0 & 0 & 0 \end{vmatrix}$$

Evaluate the following determinants:

10)
$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

11)
$$\begin{vmatrix} 1 & x & x^2 & x^3 \\ 1 & y & y^2 & y^3 \\ 1 & z & z^2 & z^3 \\ 1 & t & t^2 & t^3 \end{vmatrix}$$

12)
$$\det \begin{bmatrix} 1 & 1 & 1 & 1 & k \\ 1 & 1 & 1 & k & 1 \\ 1 & 1 & k & 1 & 1 \\ 1 & k & 1 & 1 & 1 \\ k & 1 & 1 & 1 & 1 \end{bmatrix}$$

Linear Algebra Workbook

Evaluate the determinant $\det(A)$, where $A = (a_{ij})$ is the $n \times n$ matrix, given by:

$$13) a_{ij} = \begin{cases} 1 & i = j = 1 \\ 0 & i = j \neq 1 \\ j & i < j \\ -j & i > j \end{cases}$$

$$14) a_{ij} = \begin{cases} j & i = j + 1 \\ n & i = 1, j = n \\ 0 & \text{otherwise} \end{cases}$$

$$15) a_{ij} = \begin{cases} 1 & i + j = n + 1 \\ 0 & \text{otherwise} \end{cases}$$

$$16) a_{ij} = \begin{cases} a & i = j \\ b & \text{otherwise} \end{cases}$$

Evaluate the following determinants:

$$17) \det \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & n \end{bmatrix}$$

$$18) \det \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 3 & 3 & 3 & 3 & 3 & \cdots & 3 \\ 1 & 3 & 6 & 6 & 6 & 6 & \cdots & 6 \\ 1 & 3 & 6 & 9 & 9 & 9 & \cdots & 9 \\ 1 & 3 & 6 & 9 & 12 & 12 & \cdots & 12 \\ 1 & 3 & 6 & 9 & 12 & 15 & \cdots & 15 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 3 & 6 & 9 & 12 & 15 & \cdots & 3(n-1) \end{bmatrix}$$

19) Evaluate the determinant $\det(A_{n \times n})$, where $A_{n \times n} = (a_{ij})$ is the $n \times n$ matrix, given by:

$$a_{ij} = \begin{cases} a & i = j \\ b & i = j + 1 \\ c & j = i + 1 \\ 0 & \text{otherwise} \end{cases}$$

20) Evaluate the following:

$$\begin{vmatrix} a & b & c & d & e \\ f & g & h & i & j \\ k & l & m & n & o \\ p & q & r & s & t \\ 2a+1 & -2b & 1 & x & y \end{vmatrix} + \begin{vmatrix} a & b & c & d & e \\ f & g & h & i & j \\ k & l & m & n & o \\ p & q & r & s & t \\ -a-1 & 3b & c-1 & d-x & e-y \end{vmatrix}$$

Answer Key:

- 1) a. 0 b. 0 c. 3 d. 24 e. 44 f.104
- 2) 120
- 3) 114
- 4) 6
- 5) Proved as shown in the video.
- 6) Proved as shown in the video.
- 7) -8
- 8) 16
- 9) -36
- 10) $(b-a)(c-a)(c-b)$
- 11) $(y-x)(z-x)(t-x)(z-y)(t-y)(t-z)$
- 12) $(k-1)^4(k+4)$
- 13) $1 \cdot 2 \cdot 3 \cdot 4 \cdot \dots \cdot n = n!$
- 14) $(-1)^{n-1} n!$
- 15) $(-1)^{\frac{n(n+1)}{2}}$
- 16) $[a+(n-1)b](a-b)^{n-1}$
- 17) 1
- 18) $2 \cdot 3^{n-2} \cdot 1$
- 19) $D_n = -1 + 2^{n+1}$
- 20) 0

More Rules of Determinants

Questions:

- 1) Given that A and B are 3×3 matrices satisfying $|A|=4, |B|=2$.
Evaluate the following determinants:
 - a. $|ABA^{-1}B^T|$
 - b. $|4A^2B^3|$
 - c. $| -A^{-2}B^T A^3 |$
 - d. $| -2A^2 A^T adj B |$
- 2) Given: $(PQ)^{-1}APQ = B$. Prove: $|A|=|B|$.
- 3) Let A and B be invertible matrices of order 4, such that $2AB + 3I = 0, |A|=2$.
Compute $|B|$.
- 4) Let A and B be invertible matrices of order 3, such that $A + 3B = 0, B^2 - 2A^{-1} = 0$.
Compute: $|A|, |B|$.
- 5) Prove:
 - a. $|A^{-1}| = \frac{1}{|A|}$
 - b. $|adj(A_{n \times n})| = |A|^{n-1}$
- 6) Let A be an antisymmetric matrix of odd order. Prove that $|A|=0$.
- 7) Given: A, B are matrices of order n, B is invertible, $|A|=128, 2AB = B^T A^2$. Find n .
- 8) Given: $\det(A_{n \times n}) = 2, \det(B_{n \times n}) = \frac{1}{3}$. Compute: $\det\left(\frac{1}{3}B^{-n}A^{2n}\right)$.

Answer Key:

- 1) a. 4 b. 2^{13} c. -8 d. -2^{11}
- 2) Proved as shown in the video.
- 3) $|B| = \frac{81}{32}$
- 4) $y = -\frac{2}{3} = |B|, x = -27 \cdot \left(-\frac{2}{3}\right) = 18 = |A|$
- 5)-6) Proved as shown in the video.
- 7) $n = 7$
- 8) 4^n

Cramer's rule

Questions:

Solve, using Cramer's Rule, the systems:

$$1) \begin{cases} x+2y=5 \\ 3x+4y=11 \end{cases} \quad 2) \begin{cases} x+z=3 \\ 4x+y+8z=21 \\ 2x+3z=8 \end{cases} \quad 3) \begin{cases} x+2z+5t=8 \\ -2x-6y=-8 \\ 5x+3y-7z+4t=5 \\ 2x+5y+44z=51 \end{cases}$$

$$4) \text{ Consider the following system of equations: } \begin{cases} kx+y+z+t+r=1 \\ x+ky+z+t+r=1 \\ x+y+kz+t+r=1 \\ x+y+z+kt+r=1 \\ x+y+z+t+kr=1 \end{cases}$$

- For which values of k does the system have a unique solution?
- For which value of k does the system have a unique solution with $x = \frac{1}{2}$?
- Is there a value of k such that the system has a unique solution with $x = \frac{1}{5}$?
- Prove that if the system has a unique solution, then $x = y = z = t = r$.

Answer Key:

- $x=1, y=2$
- $x=1, y=1, z=2$
- $x=1, y=1, z=1, t=1$
- The system has a unique solution only if $k \neq 1, k \neq -4$
 - $k = -2$
 - No value of k
 - Proved as shown in the video

The Adjoint Matrix

Questions:

1) Given the 2×2 matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Compute $\text{adj}(A)$ and use it to compute A^{-1} .

2) Given the 3×3 matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & -1 \\ 5 & 2 & 3 \end{bmatrix}$. Compute $\text{adj}(A)$ and use it to compute A^{-1} .

3) Given the 4×4 matrix $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. Compute $\text{adj}(A)$ and use it to compute A^{-1} .

4) Given the 5×5 matrix $A = \begin{bmatrix} -9 & 26 & -1 & 14 & 10 \\ 13 & -7 & 87 & 4 & 0 \\ 71 & 35 & 3 & 0 & 0 \\ 17 & 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Compute $\text{adj}(A)_{1,5}$, and use it to compute $(A^{-1})_{1,5}$.

5) Solve the following:

- Let A be a square matrix with $\det(A) = 1$. Prove that if all the elements of A are integers, then the elements of A^{-1} are integers too.
- Let A be an invertible lower-triangular matrix. Prove that A^{-1} is also lower-triangular.
- Prove that if A is invertible then so are $\text{adj}(A)$ and A^T .
- If matrices A, B are invertible but C, D aren't, which of these is invertible:
 - $C + D$
 - $A + B$
 - AD
 - CD
 - AB

6) Find the values of k for which the following matrix is non-invertible:

$$\begin{bmatrix} 4 & 0 & 7 & 5 & 0 \\ 0 & 0 & 3k & 0 & 0 \\ -7k^2 & 2 & 4k & k & 9+k \\ 3 & 0 & 4 & 2 & -1 \\ -5 & 0 & -8 & -3 & 2 \end{bmatrix}$$

Answer Key:

1) $\begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$

2) 1

3) -1

4) $\frac{1}{2}$

5) a-c. Proved as shown in the video.

d. AB

6) The matrix is non-invertible if and only if $k = 0$.

Geometrical Applications of Determinants

Questions:

- 1) Answer the following:
- Compute the area of the parallelograms whose vertices are given below:
 - $(0,0), (5,2), (6,5), (11,6)$
 - $(-1,0), (0,5), (1,-4), (2,1)$
 - Compute the volume of the tetrahedron with vertices:
 $(0,0,0), (1,0,-2), (1,2,4), (7,1,0)$
 - Find the equation of the plane passing through the points:
 $(3,3,-2), (-1,3,1), (1,1,-1)$
 - Compute the area of the triangle with vertices: $(1,2), (3,4), (5,8)$.

Answer Key:

- 1) a. i. 13; ii. 14
b. 22
c. $3x - y + 4z + 2 = 0$
d. 2