proprep™

Workbook

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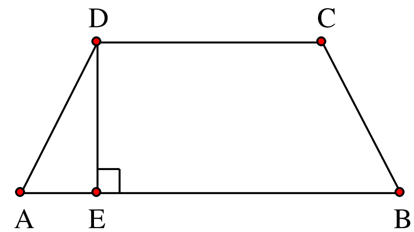
Extrema Word Problems

Geometrical Problems

Questions:

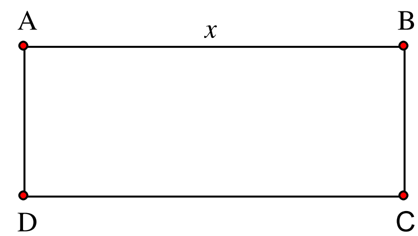
1) In an isosceles trapezoid ABCD, the length of the leg is 4cm and the length of the short base is 4cm.
DE is an altitude (See figure).
Find the length of the segment AE such that the area of the trapezoid is maximal.



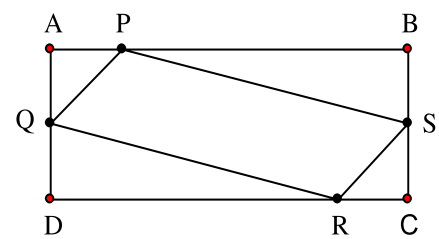
2) Given a rectangle ABCD.
Let x denote one of its sides (See figure).

a. If the circumference of the rectangle is 60cm, express the area of the rectangle in terms of x.

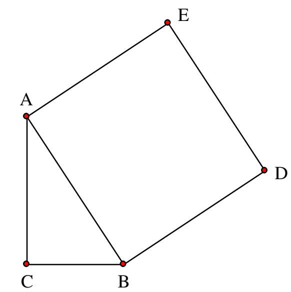
b. If the circumference of the rectangle is p, find (in terms of p) the dimensions of the rectangle with maximal area.



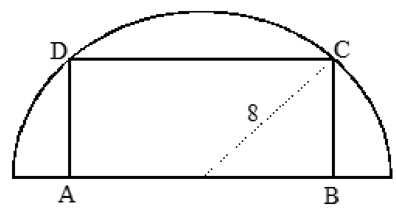
3) Given a rectangle ABCD with AD = BC = 5cm and AB = CD = 10cm.
On the sides of the rectangle, segments AP=AQ=CS=CR= x are laid off (see figure).
Which value of x gives the maximal area of parallelogram PQRS?



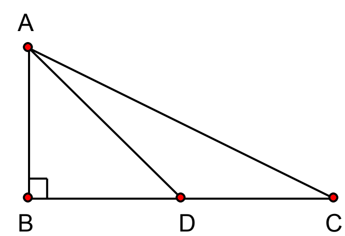
4) In a right triangle ΔABC $(\measuredangle C = 90^\circ)$ the sum of the lengths of the legs is 8cm.
On the hypotenuse AB a square ABDE is constructed.
What should the lengths of the legs be, in order to minimize the area of the pentagon AEDBC?



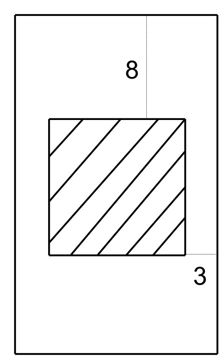
5) A rectangle ABCD is inscribed in a semicircle of radius 8cm, such that the side AB lies on the diameter and the vertices C and D lie on the circumference (see figure).
What should the length of the side AB be in order to maximize the area of the rectangle?



6) In a right triangle ΔABC $(\measuredangle B = 90^\circ)$ the sum of the lengths of the legs is 30cm.
AD is the median to the leg BC (see figure).
What should the lengths of the legs be, in order to minimize the square of the length of the median?



7) In an advertising brochure, the area of each page is 600cm^2.
The width of the margins on either side is 3cm.
Find the length and width of each page in order to maximize the printing area (shaded in the figure).



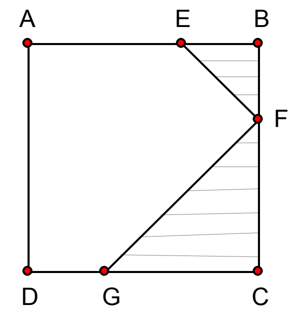
8) In square ABCD the points E, F, G are on the sides AB, CB, DC respectively, such that BE = BF, CF = CG (see figure). The side of the square is given to be 6cm.

a. Let x denote BE = BF. Express, in terms of x, the sum of the areas of the triangle EBF and FCG (shaded in the figure).

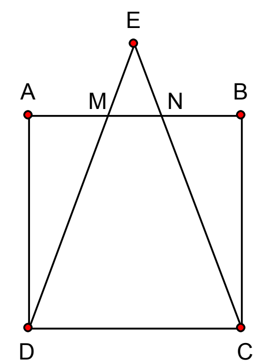
b. Answer the following:

i. Find x for which the sum of the areas of the triangles is minimal.

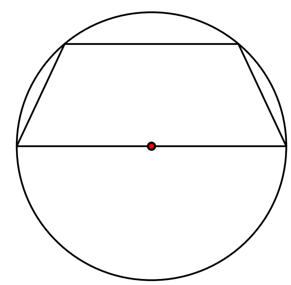
ii. Find the minimal sum of the areas of the triangles.



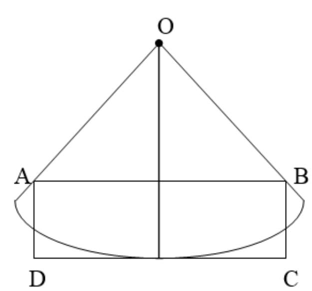
9) Given a square ABCD with side 10cm.
E is some point outside the square, such that the triangle DEC is isosceles (ED = EC). The legs of the triangle cut the side AB at points M and N (see figure).
What should the length of AM be, in order to minimize the sum of the areas of the triangles EMN, AMD, BNC?



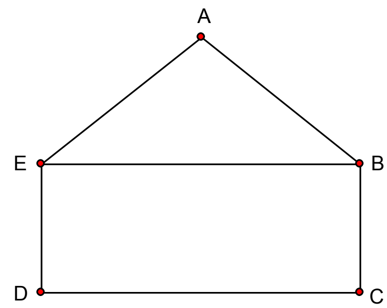
10) Given a circle of radius R.
Inscribed in this circle is an isosceles trapezoid whose long base is a diameter of the circle (see figure).
Among all such trapezoids, express in terms of R the length of the short base of the trapezoid with maximal area.



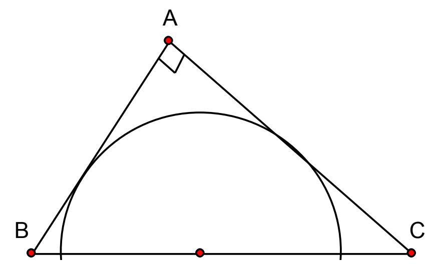
11) Given a sector which is a quarter of a circle with center O and radius 10cm. A rectangle ABCD is constructed such that the arc of the sector is tangent to the DC at its midpoint, and the points A and B are on the bounding radii (see figure).
Among all rectangles formed this way, find the smallest diagonal length.



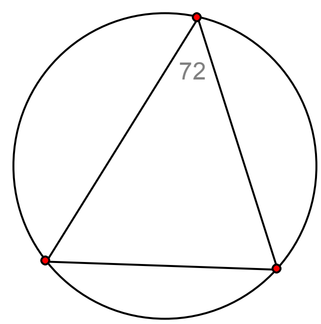
12) ABCDE is a pentagon consisting of a triangle ABE and a rectangle EBCD (see figure).
Given: BC = 2cm , AB = AE = 4cm .
Find the greatest area possible of such a pentagon.



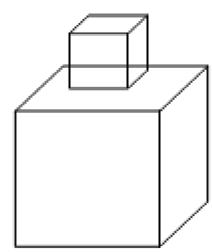
13) Consider all right-angled triangles ABC which bound a semicircle with radius R , as shown in the figure.
What are the angles of such a triangle with the least sum of its legs?



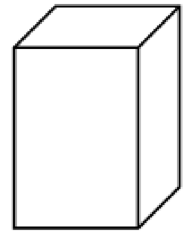
14) Consider all triangles circumscribed by a circle of radius R such that one of the angles is $\frac{2\pi}{5}$ $(72°)$, as in the figure.
Find the other angles of such a triangle with the largest circumference.



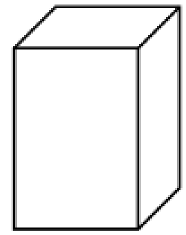
15) The height of a "tower" built from two cubes (not necessarily equal) is 8cm (See figure).
What should the length of side of the lower cube be, for the volume of the tower (sum of the volumes of the cubes) to be minimal?



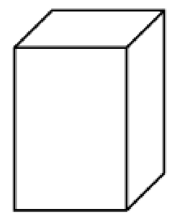
16) A box is to be built with height y-cm and a square base with side x-cm, such that the circumference of each lateral face is 12cm .
What should the value of x be, in order that the volume of the box be maximal?



17) A box with an <u>open top</u> is to be constructed, with square base and surface area (base + 4 sides) of 75cm^2.
From all such boxes, find the dimensions of the box (side of base + height) with maximal volume.

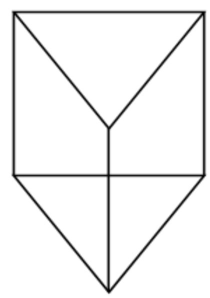


18) A skeleton (wire frame) of a box is to be constructed, with a square base and a volume of 1000cm^3.
What is the minimal length of wire needed to construct such a box?

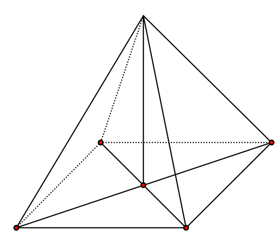


19) From a wire of length a-cm, a right triangular prism is to be constructed, whose base is an equilateral triangle.
Which part of the length of the wire should be laid off for the base x, and which part for the height y, in order that:

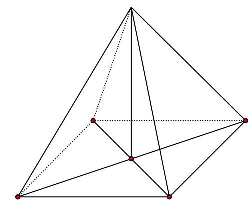
c. The surface area of the prism be minimal?
d. The volume of the prism be maximal?



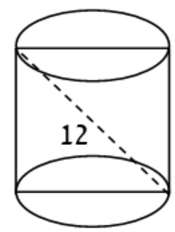
20) From all the right regular pyramids with square base and lateral edge a-cm (see figure).
Find the volume of the pyramid whose volume is maximal.



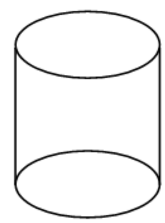
21) From all the right regular pyramids with square base and a surface area of 200cm^2, compute the volume of the pyramid whose volume is maximal.



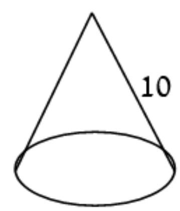
22) The diagonal length of the axial cross section of a right cylinder is 12cm (see figure). Find the height of the cylinder and its base radius, so that its volume is maximal.



23) Given a cylindrical container, <u>open on top</u>, and capacity 64m^3. The container is made entirely of tin. Show that the area of it is minimal, when the base radius is $\frac{4}{\sqrt[3]{\pi}}$ meters.



24) Among all right cones with whose generator is 10cm (see figure). What is the volume of the cone whose volume is maximal?



Answer Key:

1) $AE = \frac{-3+\sqrt{41}}{2}$

2) a. $y = x(30-x)$ b. $15x15$

3) $x = 3.75$

4) 4cm , 4cm

5) $AB = 2\sqrt{32}$

6) $BD = 24,\ AB = 6$

7) 15 cm, 40 cm

8) a. $y = x^2 - 6x + 18$ b.(i) $x = 3$cm

b.(ii) 9 cm^2

9) $x = \sqrt{12.5}$

10) $x = R$

11) $\sqrt{80}$

12) $12\sqrt{3}$

13) $90°, 45°, 45°$

14) $54°, 54°$

15) $x = 4$

16) $x = 4$

17) 5 , 2.5

18) $l = 120$

19) a. $\frac{6-\sqrt{3}}{3(12-\sqrt{3})}a$ b. $x = \frac{a}{9}$

20) $V = \frac{4}{9\sqrt{3}}a^3$

21) $V = \frac{500}{3}$

22) $x = \sqrt{24}$, $y = 4\sqrt{3}$

23) $x = \frac{4}{\sqrt[3]{\pi}}$

24) $V \approx 403.1$ cm^3

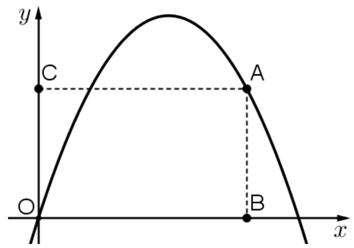
Functions and Graphs Problems

Questions:

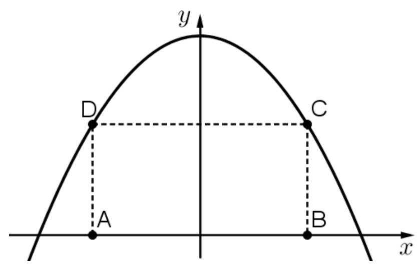
1) From point A, which is on the graph of the function $y=-x^2+5x$, perpendiculars are dropped to the axes, thus forming a rectangle ABOC (see figure).

a. What should the coordinates of A be in order to maximize the perimeter of the rectangle?

b. What should the coordinates of A be in order to minimize the perimeter of the rectangle?



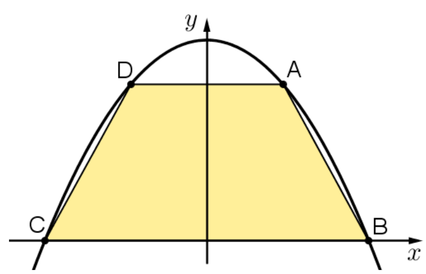
2) In parabola $y=9-x^2$ a rectangle ABCD is bounded, such that the side AB lies on the x-axis (see figure).
What should the length of side CD be in order to maximize the area of the rectangle?



3) The trapezoid ABCD is bounded by the graph of the parabola $y=9-x^2$ and the x-axis (see figure).

a. What should the coordinates of point A be in order to maximize the area of the trapezoid ABCD?

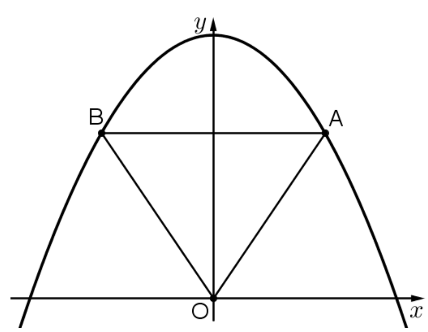
b. Compute the maximal area of the trapezoid ABCD.



4) The parabola $y=-x^2+12$ is cut by a line parallel to the x-axis at points A and B (see figure). These points are connected to the origin O, thus forming a triangle AOB.

a. What should the length of segment AB be in order to maximize the area of this triangle?

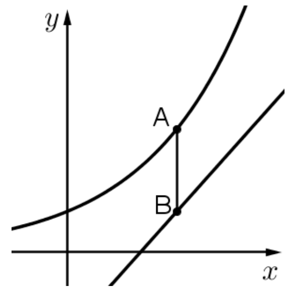
b. What is the maximum area of the triangle?



5) The figure shows the graphs of the function $y=e^x$ and of the line $y=e\cdot x-2$. A line parallel to the y-axis cuts the graphs at points A and B respectively.

a. For which value(s) of x will the length of segment AB be minimal?

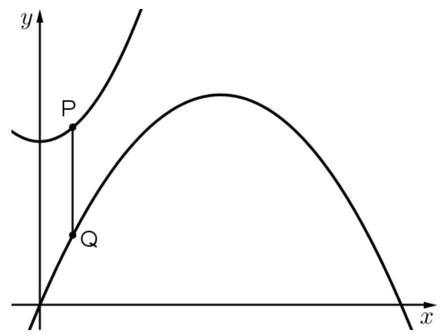
b. Is there a value of x for which this length is maximal?



6) The figure shows the graphs of two parabolas:

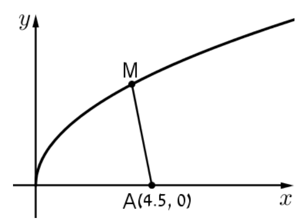
$y=-\frac{1}{4}x^2+3x,\ y=\frac{1}{2}x^2+7$.

A line parallel to the y-axis cuts the parabolas at points Q and P respectively. From all possible such segments PQ find the length of the one with minimal length.



7) Given the graph of the function $y=\sqrt{x}$ and the point $\mathrm{A}(4.5,0)$ on the x-axis (see figure).

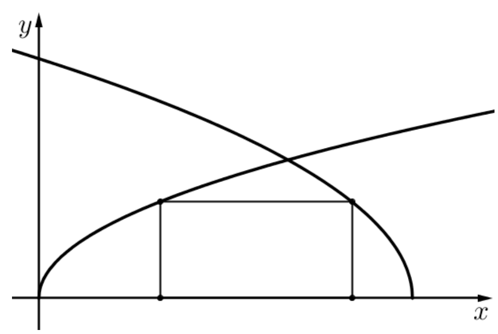
Find a point M on the graph, such that the square of the distance AM is minimal.



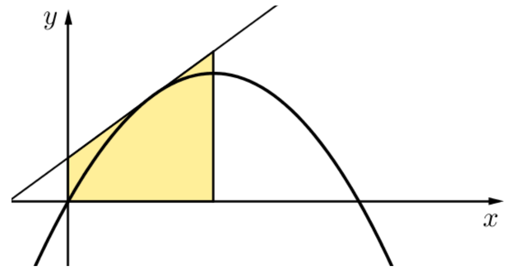
8) On the line $f(x)=3x-4$, find the point closest to the point $(0,1)$.

9) The figure shows the graphs of the functions: $f(x)=\sqrt{3x}$ and $g(x)=\sqrt{36-6x}$. A rectangle, as shown, is bounded by the graphs of the functions and by the x-axis.

Find the largest possible area of such a rectangle.

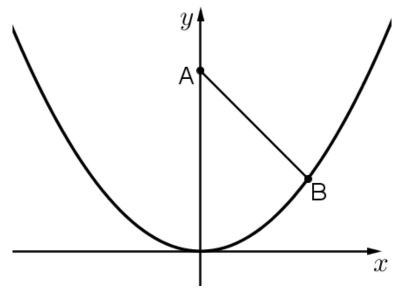


10) Though which point on the parabola $y=-x^2+2x$ should a tangent be drawn, in order to minimize the area of the trapezoid (shaded in the figure) formed by the tangent and by the lines $x=1, x=0$ and $y=0$.



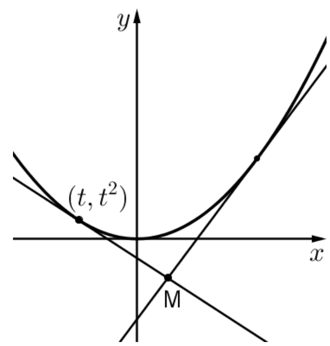
11) The point B is on the graph of the function $y=x^2$ in the first quadrant. A is the point $(0,a)$ where $a>0.5$ (see figure).

a. Express, in terms of a the coordinates of the point B for which the distance AB is minimal.
b. For which value of a is this minimal distance equal to 2?



12) Given the parabola $y=x^2$ and a tangent to it whose equation is $y=6x-9$. An additional tangent is drawn to the parabola, at the point (t,t^2) on it. The two tangents intersect at point M (see figure).

a. Express the equation of the additional tangent by means of t.
b. Find the value of t which minimizes the distance from M to the vertex of the parabola.



13) In the coordinate plane are given two points, $\mathrm{A}(2,2)$ and $\mathrm{B}(2,-2)$. The origin is at the point O. M is a point on the positive x-axis. What should the coordinates of M be in order to minimize the sum: OM+MA+MB?

Answer Key:

1) a. $A(3,6)$ b. $\varnothing$ **2)** $2\sqrt{3}$ **3)** a. $A(1,8)$ b. 32.

4) a. 4 b. 16 **5)** a. $x=1$ b. No. **6)** 4 **7)** $M(4,2)$

8) $(1.5,0.5)$ **9)** 8 **10)** $(0.5,0.75)$

11) a. $B\left(\sqrt{\frac{a-1}{2}},\frac{2a-1}{2}\right)$ b. $\frac{17}{4}$ **12)** a. $y=2tx-t^2$ b. $-\frac{3}{37}$ **13)** $M(0.845,0)$

Business Applications Problems

Questions:

1) When a dairy company sells x gallons of chocolate milk a day, it can get a price of $P(x) = -\frac{1}{4}x + 10$ dollars per gallon.

a. What is the price per gallon if 4 gallons a day are sold?
b. What is the price per gallon if 12 gallons are sold?
c. How many gallons a day are sold if the price is $\$6$ per gallon?
d. Sketch the graph of the demand function and fins its domain.
e. The given demand function describes the price as a function of the quantity sold. Change the form of the function so that it describes the quantity sold as a function of the price.

2) The demand function of a certain product is $P(x) = -0.6x + 120$ (in dollars).

a. Find the total revenue function and its domain.
b. If $x = 20$, what is the price of the product and what is the revenue?
c. If the price is $\$12$, what is the revenue?

3) The total revenue function of a certain product is $R(x) = -0.08x^2 + 40x$ (in dollars).

a. What is the domain of the revenue function?
b. Sketch the graph of the revenue function.
c. Find the demand function and sketch its graph.

4) The demand function of a certain product is $P(x) = -0.4x + 100$ dollars per unit.

a. Find the domain of the function.
b. Find the total revenue function and the average revenue function.
c. Find the marginal revenue function.
d. Which value of x gives the maximal total revenue, and what is it?

5) The demand function of a certain product is $P(x) = -6x^2 + 240x + 1800$ (dollars).

a. Find the total revenue function and the marginal revenue function.
b. If $x = 40$, is it worthwhile to increase production?
c. Which value of x gives the maximal total revenue, and what is it?

6) The demand function of a certain product is $Q(x)=10x-\frac{x^2}{5}$, where Q is the quantity sold, and x is the price in dollars.

a. Find the unit price which gives the maximal total revenue.
b. What is the demand (Q) in this case?
c. What is the marginal demand at the price found above? What does it mean?

7) A manufacturer produces x lbs. of coffee a day at a total cost of $C(x)=5x+150$ (in $\$$).

a. Sketch a graph of the total cost function. What are the fixed costs?
b. How many lbs. of coffee does the manufacturer produce if the total cost is $\$1,000$?
c. What is the total cost producing 20 lbs. of coffee?
d. Find the marginal cost function.

8) The total cost function of a hat manufacturer is $TC(x)=0.04x^2+10x+400$ ($/day).

a. Compute the average cost for producing 40 hats per day.
b. How many hats should be produced per day to minimize the average cost?
c. Compute the marginal cost for producing 100 hats per day.
   What can we conclude?

9) The total cost function of a certain product is $C(x)=0.004x^2+10x+200$.

a. Compute the cost when $x=100$ and when $x=101$.
b. Compute the marginal cost when $x=100$.
c. Use (a) to compute the cost increase when x increases from 100 to 101 and compare this to the result of (b). What can we conclude?
d. Is the rate-of-change of the cost function increasing or decreasing?

10) A manufacturer has a product with demand function $P(x)=100-0.06x$ dollars/unit, and a total cost function of $TC(x)=200+4x$.

a. Find the amount x that he need to produce in order to maximize his profit.
b. What is the maximum profit?

11) A manufacturer has a product with a constant demand function $P(x)=20$, and a total cost function $TC(x)=300+2x^2$ (both in dollars).

a. Find the quantity x he needs to produce in order to maximize hos profit.
b. In this case, what is the maximum profit?

12) A manufacturer has a product with demand function $P(x) = -0.15x + 50$, and a marginal cost function $MC(x) = 0.06x^2 + 20$ (both in $\$$).
What's the quantity x he needs to produce in order to maximize profit?

13) A manufacturer has a product with demand function $x = \frac{5000 - 50P}{3}$, and a total cost function $TC(x) = 200 + 4x$ (both in $\$$).

a. Find the quantity x he needs to produce in order to maximize his profit.
b. In this case, what is the maximum profit?

14) A manufacturer has a product with marginal cost function $MC(x) = 0.06x^2 + 20$ (in $\$$).
Find the total cost function, given that when $x = 10$ the total cost is $\$225$.

Answer Key:

1) a. $9 b. $7 c. 16 gallon

d. e. $x(p)=40-4P$

2) a. $R(x)=-0.6x^2+120x$, $x\geq 0$ b. Price: $\$108$; revenue: $\$2160$ c. $\$2160$

3) a. $x\geq 0$ c. $P(x)=-0.08x+40$

4) a. $x\geq 0$ b. $TR(x)=-0.4x^2+100x$ $AR(x)=-0.4x+100$

c. $MR(x)=-0.8x+100$ d. $x_v=125$ $\$6,250$

5) a. $R(x)=-6x^3+240x^2+18$ $MR(x)=-18x^2+480x+1800$ b. No

c. $x=30$ $R(x)=\$108,000$

6) a. $\$33\frac{1}{3}$ b. $111\frac{1}{9}$ c. $-3\frac{1}{3}$

7) a. $150 b. 170 lbs. c. $250

d. $5

8) a. $AC(x)=\$21.60$ b. 100 hats

c. $\$18$; TC increases by : $\$18$

9) a. $C(100)=\$1,240$ $C(101)=\$1,205.804$ b. $MC(100)=\$10.8$

c. $C(101)-C(100)=\$10.804$ The marginal cost $x=100$ is roughly equal to the cost of producing the 101^{th} item.

10) a. 800 units b. $\$38,200$

11) a. 5 units b. $\$-250$

12) $x=20$

13) a. 800 units b. $\$38,200$

14) $TC(x)=0.02x^3+20x+5$

Pricing Problems

Questions:

1) A washing machine manufacturer sells 500 machines a week at a $\$225$ each. The production cost of each machine is $\$125$. A market survey shows, that for each $\$5$ reduction in price, the number of machines sold per week increases by 50.

a. What should the manufacturer charge per machine in order to maximize profit?
b. What are his production costs in this case? Are they necessarily minimal?

2) A global cell phone company offers packages of 200 minutes of international air time a month for $\$100$, prorated accordingly (i.e. $50¢$/min. above 200 minutes). A market survey showed, that for each $\$2$ reduction in rate, customers will (on average) talk 10 more minutes a month. E.g. 210 mins for a $\$98$ package.

a. Find the package price which will maximize revenue.
b. At this price, how many minutes a month will costumers (on average) talk?

3) An artist creates pieces of jewelry, at a production cost of $\$30$ each. When he charges $\$40$ each, he sells 100 pieces a day. For each $\$2$ price increase, he sells 4 a day.

a. How many pieces a day should he make in order to maximize his profit?
b. At what price (per piece) would he sell this quantity?
c. What then would be his total daily production cost for the jewelry?

4) The "*Good Trip*" company rents coaches to groups of 30 tourists at a price of $\$100$ per person. For each additional tourist, the group gets a price discount of $\$2$ each. What number of tourists in the group will yield maximum profit for the company? Assume that costs are fixed.

5) A cell phone company charges $10¢$ to send an SMS and the average customer sends 200 a month. For every $1¢$ raise in the price, the average number sent drops by 10. What should the company charge for an SMS in order to maximize its revenue?

6) The "*Rialto*" theatre sells 60 tickets weekly at $\$9$ a ticket. Each price reduction of $10¢$ results in 2 more ticket sales a week. What should the price of a ticket be in order to maximize the theater's weekly revenue? What is this maximal revenue?

7) The production of a "*Spongebob Squarepants*" doll costs "*Nickelodeon*" company $\$5$ to produce. If the company sells the doll for $\$9$, it can sell 200 units a day. For each $10¢$ reduction in price, it can sell an additional 10 dolls a day.
What is the maximum profit the company makes on the sale of these dolls?

8) The "*Office Depot*" company purchases a number of equally priced items for a total of $\$800$. It sells 5 of the times at a 20% profit and the remaining items at a markup of $\$2$ each. Prove that the profit the company makes on this deal is at least $\$70$.

9) The BMX bike company sells 300 bicycles of a certain model, weekly, at a price of $\$100$ apiece. For each x additional bicycle it sells, it lowers the price by $0.4x$ dollars and the price of the first 300 is lower by only $0.2x$ dollars each.
[*The price is lowered by mailing a rebate check to the customer*]
How many bicycles does the company need to sell in order to maximize its revenue?

Answer Key:

1) a. $\$200$ b. $\$93,750$; Not minimal

2) a. $\$70$ b. 350 mins

3) a. 60 pieces b. $\$60$ c. $\$1800$

4) 40 tourists

5) $15¢$

6) Ticket price: $\$6$; maximal revenue: $\$720$

7) $\$900$

8) Refer to the video

9) 350 bikes