proprep

Workbook

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Definite Integrals

Definite Integrals

Questions

Compute the following integrals:

1) $\int_1^4 \left(2x^2 - 4x + 1\right) dx$

2) $\int_0^2 \frac{2x+1}{x^2+x+1} dx$

3) $\int_2^3 xe^{-x} dx$

4) $\int_1^4 \frac{(\ln x)^4}{x} dx$

5) $\int_1^\pi \cos^2 4x dx$

6) $\int_0^4 f(x) \ ; \ f(x) = \begin{cases} \sqrt{x} & 0 \le x \le 1 \\ \frac{1}{x^2} & x \ge 1 \end{cases}$

7) $\int_{-1}^4 \sqrt{4 + |x-1|} dx$

8) $\int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx$

9) $\int_0^{\pi/2} \frac{\sqrt[4]{\sin x}}{\sqrt[4]{\sin x} + \sqrt[4]{\cos x}} dx$

10) Let f be a continuous function. Prove that:

a. If f is even then $\int_{-a}^a f(x) dx = 2\int_0^a f(x) dx$.

b. If f is odd then $\int_{-a}^a f(x) dx = 0$.

11) Compute the following integrals:

a. $\int_{-4}^4 \frac{\cos x}{x^3 + x^5} dx$.

b. $\int_{-1}^1 \frac{\sin x + 1}{x^2 + 1}$.

Answer Key

1) 13

2) $\ln 7$

3) $-4e^{-3}+3e^{-2}$

4) $\frac{1}{5}\ln^5 4$

5) $\frac{1}{2}\left(\pi-1-\frac{1}{8}\sin 8\right)=1.062$

6) $1\frac{5}{12}$

7) $\frac{2}{3}\left(-16+6^{1.5}+7^{1.5}\right)=11.478$

8) $\frac{\pi^2}{4}$

9) $\frac{\pi}{4}=0.785$

10) Refer to the video

11) $\frac{\pi}{2}$

Inequalities

Questions

1) Prove: $\frac{2}{41} \le \int_{-1}^{3} \frac{dx}{1+x} \le 4$.

2) Prove: $6 \le \int_{-4}^{2} \sqrt{1+x^2}\,dx \le 6\sqrt{17}$.

3) Prove: $2 \le \int_{0}^{2} e^{x^2}\,dx \le 2e^4$.

4) Prove: $\frac{1}{2}e^{-10} < \int_{0}^{10} \frac{e^{-x}}{x+10}\,dx < 1$.

5) Prove: $\frac{\pi}{14} < \int_{0}^{\pi/2} \frac{dx}{3+4\sin^2 x}\,dx \le \frac{\pi}{6}$.

6) Compute the integral: $-\frac{1}{2} \le \int_{0}^{1} x\sin\left(\frac{\ln x}{x+1}\right)dx \le \frac{1}{2}$.

7) Prove: $\int_{0}^{\pi} x^2 \arctan\left(\frac{\sin x}{x+4}\right)dx \le \frac{\pi^4}{6}$.

Answer Key

To view the answers to the exercises, please refer to the appropriate videos on site.

Further Exercises - Criterion for Integrability

Questions

1) Let $f:[a,b]\to\mathbb{R}$ be a bounded function. Suppose that there is a partition P of $[a,b]$ such that $L(P,f)=U(P,f)$. Show that *f* is a constant function.

2) In each of the following cases, evaluate the upper and lower integrals of *f* and show that *f* is integrable. Find the integral of *f* .

a. For $\alpha\in\mathbb{R}$, define $f:[a,b]\to\mathbb{R}$ by $f(x)=\alpha$ for every $x\in[a,b]$.

b. $f(x)=0$ for $0\le x<\frac{1}{2}$, $f(\frac{1}{2})=10$ and $f(x)=1$ for $\frac{1}{2}<x\le 1$.

c. $f(x)=x$ for all $x\in[0,1]$.

3) Let $f:[a,b]\to\mathbb{R}$ be bounded and (P_n) be partitions such that $U(P_n,f)-L(P_n,f)\to 0$.

a. Show that f is integrable.

b. Show that $\lim_{n\to\infty}U(P_n,f)=\lim_{n\to\infty}L(P_n,f)=\int_a^b f(x)dx$.

4) In each of the following cases, show that *f* is integrable using the Riemann criterion.

a. $f(x)=x$ on $[0,1]$.

b. $f(x)=x^2$ on $[0,1]$.

c. $f(x)=\frac{1}{x}$ on $[1,2]$.

5) Let f_1, f_2, f be bounded functions on [0,1] such that $f_1(x)\le f(x)\le f_2(x)$ for all $x\in[0,1]$.

Suppose that f_1 and f_2 are integrable and $\int_0^1 f_1(x)dx=\int_0^1 f_2(x)dx=I$.

Show that f is integrable and find $\int_0^1 f(x)dx$.

6) Let $f:[0,1]\to\mathbb{R}$ be such that $f(x)=x$ for *x* rational and $f(x)=0$ for *x* irrational. Evaluate the upper and lower integrals of *f* and show that *f* is not integrable.

7) Let $f:[0,1]\to[0,1]$ be such that

$$f(x)=\begin{cases}\frac{1}{q} & \text{if } x=\frac{p}{q}\neq 1 \text{ where } p,q\in\mathbb{N} \text{ and } p,q \text{ have no common factors}\\ 0 & \text{if } x \text{ is irrational or } x=0 \text{ or } x=1\end{cases}$$

a. For any $N\in\mathbb{N}$ consider the set
$A_N=\left\{x\in(0,1): x=\frac{p}{q} \text{ where } p,q\in\mathbb{N},\ q\le N \text{ and } p,q \text{ have no common facrors}\right\}$.
Show that the set A_N is finite.

b. For given $N\in\mathbb{N}$ and $\varepsilon>0$ show that there are intervals $[x_1,x_2],[x_3,x_4],\dots,[x_{2m-1},x_{2m}]$ such that $0<x_1<x_2<x_3<x_4<\dots<x_{2m-1}<x_{2m}<1$,
$A_N\subseteq(x_1,x_2)\cup(x_3,x_4)\cup\dots\cup(x_{2m-1},x_{2m})$ and $|x_1-x_2|+|x_3-x_4|+\dots+|x_{2m-1}-x_{2m}|\le\frac{\varepsilon}{2}$.

c. Show that f is integrable.

d. Find two integrable functions *g* and *h* on [0,1] such that $g\circ h$ (composition) is not integrable.

8) Let $f:[a,b]\to\mathbb{R}$ be integrable and $[c,d]\subseteq[a,b]$. Show that *f* is integrable on $[c,d]$.

9) a. Let f be bounded on $[c,d]$, $M=\sup\{f(x):x\in[c,d]\}$, $M'=\sup\{|f(x)|:x\in[c,d]\}$, $m=\inf\{f(x):x\in[c,d]\}$ and $m'=\inf\{|f(x)|:x\in[c,d]\}$. Show that $M'-m'\le M-m$.

b. Let $f:[a,b]\to\mathbb{R}$ be integrable. Show that $|f|$ and f^2 are integrable.

10) Find an example of $f:[0,1]\to\mathbb{R}$ such that

a. $|f|$ is integrable but f is not integrable.

b. f^2 is integrable but f is not integrable.

11) Let f and g be two integrable functions on $[a,b]$.

a. Show that if $f(x)\le g(x)$ for all $x\in[a,b]$ then $\int_a^b f(x)dx\le\int_a^b g(x)dx$.

b. Show that $\left|\int_a^b f(x)dx\right|\le\int_a^b|f(x)|dx$

c. If $m\le f(x)\le M$ for all $x\in[a,b]$, show that $m(b-a)\le\int_a^b f(x)dx\le M(b-a)$.

Use this inequality to show that $\frac{\sqrt{3}}{8}\le\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\sin x}{x}dx\le\frac{\sqrt{2}}{6}$.

12) Let $f:[a,b]\to\mathbb{R}_{\geq 0}$ (meaning $f(x)\geq 0$).

a. If f is continuous and $\int_a^b f(x)dx=0$, show that $f(x)=0$ for all $x\in[a,b]$.

b. Give an example where f is integrable on $[a,b]$ and $\int_a^b f(x)dx=0$ but $f(x_0)>0$ for some $x_0\in[a,b]$. Remark: *f* will not be continuous.

13) Let $f:[0,1]\to\mathbb{R}$ be a bounded function.
Suppose that for any $c\in(0,1)$, f is integrable on $[c,1]$.

a. Show that f is integrable on $[0,1]$.

b. Use (a) to show that $f(x)=\begin{cases}0 & \text{if } x=0\\ \sin\frac{1}{x} & \text{if } x\in(0,1]\end{cases}$ is integrable on [0,1].

14) Let $f:[a,b]\to\mathbb{R}$ be a continuous function. Suppose that, whenever the product fg is integrable on $[a,b]$ for some integrable function g, we have $\int_a^b (fg)(x)dx=0$. Show that $f(x)\equiv 0$; in other words $f(x)=0\ \forall x\in[a,b]$.

15) [Cauchy-Schwarz inequality]

a. Let $x_1,x_2,...x_n,y_1,y_2,...y_n\in\mathbb{R}$. Prove that $\left|\sum_{i=1}^n x_iy_i\right|\leq\left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^n y_i^2\right)^{\frac{1}{2}}$.

Hint: $\sum_{i=1}^n (tx_i+y_i)^2\geq 0\quad\forall t\in\mathbb{R}$.

b. Let f,g be two integrable functions on $[a,b]$.

Prove that $\left|\int_a^b f(x)g(x)dx\right|\leq\left(\int_a^b f(x)^2dx\right)^{\frac{1}{2}}\left(\int_a^b g(x)^2dx\right)^{\frac{1}{2}}$.

Hint: $\int_a^b [tf(x)+g(x)]^2\,dx\geq 0\quad\forall t\in\mathbb{R}$.

16) Let $f:[a,b]\to\mathbb{R}$ be integrable. Suppose that the values of f are changed at a finite number of points. Show that the modified function is integrable.

Answer Key

To view the answers to the exercises, please refer to the appropriate videos on site.

Riemann Sum

Questions

1) Evaluate: $\lim\limits_{n\to\infty}\dfrac{1^4+2^4+3^4+\cdots+n^4}{n^5}$

2) Evaluate: $\lim\limits_{n\to\infty}\left\{\dfrac{1}{n+1}+\dfrac{1}{n+2}+\cdots\dfrac{1}{n+n}\right\}$

3) Evaluate: $\lim\limits_{n\to\infty}\left\{\dfrac{n}{n^2+1^2}+\dfrac{n}{n^2+2^2}+\cdots+\dfrac{n}{n^2+n^2}\right\}$

4) Evaluate: $\lim\limits_{n\to\infty}\left\{\dfrac{1}{\sqrt{n^2+1^2}}+\dfrac{1}{\sqrt{n^2+2^2}}+\cdots+\dfrac{1}{\sqrt{n^2+n^2}}\right\}$

5) Evaluate: $\lim\limits_{n\to\infty}\left\{\dfrac{\sqrt{n+1}+\sqrt{n+2}+\cdots\sqrt{2n}}{n^{3/2}}\right\}$

6) Evaluate: $\lim\limits_{n\to\infty}\dfrac{\sin\frac{1}{n}+\sin\frac{2}{n}+\cdots+\sin\frac{n}{n}}{n}$

7) Evaluate: $\lim\limits_{n\to\infty}\dfrac{1+\sqrt[n]{e}+\sqrt[n]{e^2}+\sqrt[n]{e^3}+\cdots\sqrt[n]{e^{n-1}}}{n}$

8) Evaluate: $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\ln\sqrt[n]{1+\dfrac{k}{n}}$

9) Use the Riemann sum definition of integral to evaluate $\int\limits_0^x x\,dx$

Hint: $1+2+3+\cdots+n=0.5n(n+1)$.

10) Use the Riemann sum definition of integral to evaluate $\int\limits_0^1 x^2\,dx$

Hint: $1^2+2^2+3^2+\cdots+n^2=\dfrac{1}{6}n(n+1)(2n+1)$

11) Use the Riemann sum definition of integral to evaluate $\int_0^1 x^3 dx$

Hint: $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4} n^2 (n+1)^2$

12) Use the Riemann sum definition of integral to evaluate $\int_0^{\pi} \sin x dx$

Hint: $\sin(\alpha) + \sin(2\alpha) + \cdots + \sin(n\alpha) = \dfrac{\sin\left(\frac{n}{2}\alpha\right)\sin\left(\frac{n+1}{2}\alpha\right)}{\sin\left(\frac{\alpha}{2}\right)}$

13) Use the Riemann sum definition of integral to evaluate $\int_2^5 (2x^2 + 3x) dx$

Hint: $1 + 2 + 3 + \cdots + n = \frac{1}{2} n(n+1)$, $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6} n(n+1)(2n+1)$

Answer Key

1) $\frac{1}{5}$	**2)** $\ln 2$	**3)** $\frac{\pi}{4}$	**4)** $\ln(1+\sqrt{2})$	**5)** 1.219
6) $1-\cos(1)$	**7)** $1-\cos(1)$	**8)** $2\ln 2 - 1$	**9)** $\frac{1}{2}$	**10)** $\frac{1}{3}$
11) $\frac{1}{4}$	**12)** 2	**13)** 109.5		

Fundamental Theorem of Calculus

Questions

1) Recall the Fundamental Theorem of Calculus: $I(x)=\int_a^x f(t)dt \Rightarrow I'(x)=f(x)$.

Prove the following generalization: $I(x)=\int_a^{b(x)} f(t)dt \Rightarrow I'(x)=f(b(x))b'(x)$.

2) We previously generalized the FTC to: $I(x)=\int_a^{b(x)} f(t)dt \Rightarrow I'(x)=f(b(x))b'(x)$

Generalize again to prove: $I(x)=\int_{a(x)}^{b(x)} f(t)dt \Rightarrow I'(x)=f(b(x))b'(x)-f(a(x))a'(x)$.

3) Differentiate the following functions:

a. $I(x)=\int_2^x e^{-t^2}dt$

b. $I(x)=\int_1^{x^3} \frac{\ln t}{t^2}dt$

c. $I(x)=\int_2^{x^3+x} t\ln t dt$

d. $I(x)=\int_{x^3}^{x^2} \frac{dt}{\sqrt{1+t^4}}$

4) Evaluate the following limits:

a. $\lim_{x\to 0}\frac{\int_0^x \frac{tdt}{\cos t}}{\sin^2 x}$

b. $\lim_{x\to 0}\frac{1}{x^3}\int_0^{x^2}\sin\sqrt{t}dt$

c. $\lim_{x\to 4}\frac{x}{x-4}\int_4^x e^{t^2}dt$

5) Investigate the function $F(x)=\int_0^x (t+1)^4(t-1)^{10}dt$, particularly the following features:

a. Domain of definition
b. Extrema and intervals of increase/decrease
c. Inflection points and intervals of concavity/convexity

Answer Key

1) Proov.
2) Proov.

3) a. $I'(x)=e^{-x^2}$ b. $\frac{9\ln x}{x^4}$ c. $I'(x)=(x^3+x)\ln(x^3+x)(3x^2+1)$

d. $I'(x)=\frac{2x}{\sqrt{1+x^8}}-\frac{3x^2}{\sqrt{1+x^{12}}}$

4) a. $\frac{1}{2}$ b. $\frac{2}{3}$ c. $4e^{16}$

5) a. All x b. No extrema, the function is always increasing

c. Inflection points: -1, $-\frac{3}{7}$, 1

concave: $x<-1$ or $-\frac{3}{7}<x<1$

convex: $-1<x<-\frac{3}{7}$ or $x>1$

Advanced exercises - Riemann Sum and FTC

Questions

1) a. Show that every continuous function on a closed bounded interval is a derivative.
b. Show that an integrable function on a closed bounded interval need not be a derivative.

2) a. Let $f:[-1,1]\to\mathbb{R}$ be defined by $f(x)=\begin{cases}0\ ; & -1\le x<0\\ 1; & 0\le x\le 1\end{cases}$ and define $F(x)=\int_{-1}^{x} f(t)dt$ for $-1\le x\le 1$. Sketch the graphs of f and F and observe that:

i. f is not continuous (at 0) but F is continuous.
ii. F is not differentiable at 0.

b. Give an example of a function $f:[-1,1]\to\mathbb{R}$ such that f is not continuous at 0 but $F(x)=\int_{-1}^{x} f(t)dt$ is differentiable at 0.

3) Let $f:[a,b]\to\mathbb{R}$ be integrable. Show that $\int_a^b f(t)dt=\lim_{x\to b^-}\int_a^x f(t)dt$.

4) Prove the second FTC by assuming the integrand to be continuous.
Let f be ~~integrable~~ continuous on $[a,b]$.
If there is a differentiable function F on $[a,b]$ such that $F'=f$ then $\int_a^b f(x)dx=F(b)-F(a)$.

5) Define $f:[-1,1]\to\mathbb{R}$ by $f(x)=\begin{cases}2x\sin\frac{1}{x^2}-\frac{2}{x}\cos\frac{1}{x^2} & ;\ x\ne 0\\ 0 & ;\ x=0\end{cases}$,

and define $F:[-1,1]\to\mathbb{R}$ by $F(x)=\begin{cases}x^2\sin\frac{1}{x^2} & ;\ x\ne 0\\ 0 & ;\ x=0\end{cases}$.

Show that $F'=f$ but $\int_{-1}^{1} f(t)dt$ doesn't exist.

6) Let $f:[0,1]\to\mathbb{R}$ be continuous such that $|f(x)|\le\int_0^x f(t)dt$ for all $x\in[0,1]$.
Show that $f(x)=0\ \forall x\in[0,1]$.

7) Let $f:\mathbb{R}\to\mathbb{R}$ be continuous. Define $g(x)=\int_0^x (x-t)f(t)dt, \forall x\in\mathbb{R}$. Show that $g''=f$.

8) Let f be continuous on $\mathbb{R}$ and $\alpha \neq 0$. Define $g(x) = \frac{1}{\alpha}\int_0^x f(t)\sin[\alpha(x-t)]dt$.

Show that $f(x) = g''(x) + \alpha^2 g(x)$.

9) Let $f:[0,1] \to \mathbb{R}$ be differentiable on [0,1].

Show that there exists $c \in (0,1)$ such that $\int_0^1 f(x)dx = f(0) + \frac{1}{2}f'(c)$.

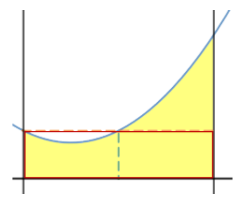
10) Let $f:[0,1] \to \mathbb{R}$ be continuous and suppose $\int_0^1 f(x)dx = 1$.

Show that there exists $c \in (0,1)$ such that $f(c) = 3c^2$.

11) Let $f:[0,\frac{\pi}{4}] \to \mathbb{R}$ be continuous. Show that $\exists c \in [0,\frac{\pi}{4}]$ such that $2\cos 2c\int_0^{\pi/4} f(t)dt = f(c)$.

12) Let $f:[0,a] \to \mathbb{R}$ be such that $f''(x) > 0$ for every $x \in [0,a]$.

Show that $\int_0^a f(x)dx > af(\frac{a}{2})$.



13) Let $f:[a,b] \to \mathbb{R}$ be continuous and suppose that $\int_a^x f(t)dt = \int_x^b f(t)dt \ \ \forall x \in [a,b]$.

Show that $f(x) = 0 \ \ \forall x \in [a,b]$.

14) Let $f,g:[a,b] \to \mathbb{R}$ be integrable and suppose that f is increasing and g is non-negative.

Show that there exists $c \in [a,b]$ such that $\int_a^b f(x)g(x)dx = f(b)\int_a^c g(x)dx + f(a)\int_c^b g(x)dx$.

15) a. Show that the Mean Value Theorem implies the First Mean Value Theorem for Integrals.
b. Show that the First Mean Value Theorem for Integrals implies the Mean Value Theorem for functions with a continuous first derivative.

16) Use the First MVT for Integrals to show that $\int_n^{n+1} \frac{1}{x}dx < \frac{1}{n}$ for all $n \in \mathbb{N}$.

17) Let $f,g:[a,b] \to \mathbb{R}$ be continuous and $\int_a^b f(x)dx = \int_a^b g(x)dx$.

Show that there exists $c \in [a,b]$ such that $f(c) = g(c)$.

18) Show that $\frac{\pi^2}{9} \leq \int_{\pi/6}^{\pi/2} \frac{x}{\sin x}dx \leq \frac{2\pi^2}{9}$.

19) Let $f:[0,1]\to\mathbb{R}$ be continuous. Show that $\lim_{n\to\infty}\int_0^1 x^n f(x)dx=0$.

20) Let $f:[0,1]\to\mathbb{R}$ be continuous. Show that $\lim_{n\to\infty}\int_0^1 f(x^n)dx=f(0)$.

21) Let $f:[a,b]\to\mathbb{R}$ be continuous. Show that $\lim_{\|P\|\to 0}S(P,f)=\int_a^b f(x)dx$. Notation:

$P=\{\overset{=a}{x_0},x_1,\ldots,\overset{=b}{x_n}\}$, $S(P,f)=\sum_{i=1}^n f(c_i)\Delta x_i$, $\Delta x_i=x_i-x_{i-1}$, $c_i\in[x_{i-1},x_i]$, $\|P\|=\max_{1\le i\le n}\{\Delta x_i\}$.

22) Let $a_n=\ln\left(\frac{(n!)^{\frac{1}{n}}}{n}\right)$ for all $n\in\mathbb{N}$. Convert a_n to a Riemann sum and find $\lim_{n\to\infty}a_n$.

23) Let $f,g:[a,b]\to\mathbb{R}$ be such that f',g' are continuous on $[a,b]$.

Show that $\int_a^b f(x)g'(x)dx=\left[f(x)g(x)\right]_a^b-\int_a^b f'(x)g(x)dx$.

24) Let $\phi:[\alpha,\beta]\to\mathbb{R}$ be continuously differentiable and let f be continuous on the range of ϕ.

Then $\int_\alpha^\beta f(\phi(t))\phi'(t)dt=\int_{\phi(\alpha)}^{\phi(\beta)} f(x)dx$. [Substitution: $x=\phi(t),\ dx=\phi'(t)dt$].

25) Let $f:[a,b]\to\mathbb{R}$ be continuous and let $u,v:[a,b]\to\mathbb{R}$ be differentiable.

If the ranges of u and v are $\subseteq[a,b]$, show that $\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)dt=f(v(x))\frac{dv}{dx}-f(u(x))\frac{du}{dx}$.

26) Define $f:[1,\infty)$ by $f(x)=\int_1^x \frac{\ln t}{1+t}dt$. Solve the equation $f(x)+f(\frac{1}{x})=2$.

27) Let the functions f, g be continuous on $[a,\infty)$ and differentiable on (a,∞).
Suppose $f(a)=g(a)$ and $f'(x)\le g'(x)$ for all $x>a$. Prove that $f(x)\le g(x)$ for all $x>a$.

Answer Key

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