

Workbook

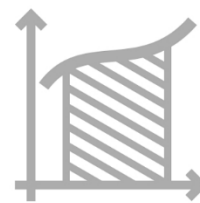


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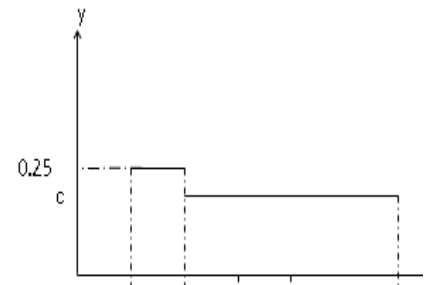
The Continuous Random Variable

General Probabilities without Integrals

Questions:

1) X is a continuous variable having the density function shown below:

- a. Find the value of c .
- b. Construct the cumulative distribution function.
- c. Calculate the following probabilities:
 - i. $P(X = 4)$
 - ii. $P(X > 1.5)$
 - iii. $P(1.5 < X < 5)$
 - iv. $P(5 < X < 10)$
- d. Find the median of X .



2) A continuous random variable X has the following density function:

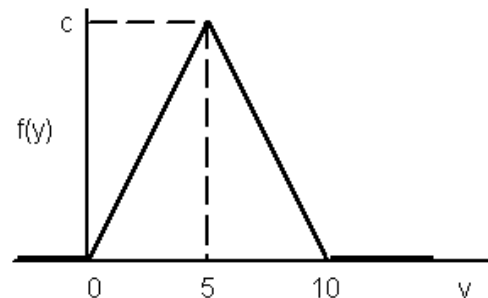
$$f(x) = cx, \text{ where } 0 \leq x \leq b, 0 \text{ otherwise.}$$

It is known that $P(0 < X < 1) = \frac{1}{4}$.

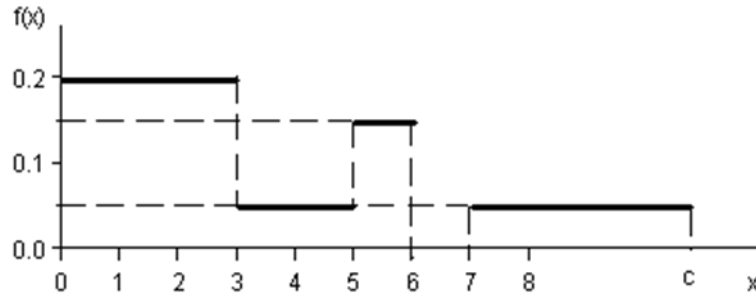
- a. Write the density function of X .
- b. Find the median of X .
- c. What are the chances of X being less than 0.5?

3) The diagram below shows the density function of the random variable Y

- a. Calculate c .
- b. Write the cumulative distribution function of Y .
- c. Calculate the probabilities: $P(Y > 4)$, $P(7.5 < X < 15.5)$, $P(Y \leq 3)$, $P(Y = 7)$.
- d. Calculate the bottom 10th percentile $Y_{0.1}$, the bottom 25th percentile $Y_{0.25}$, and the median of Y and the upper 10th percentile $Y_{0.9}$.



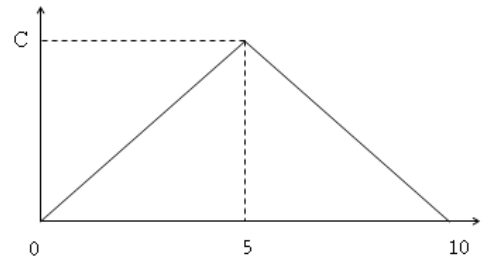
4) The diagram below shows the density function of the random variable X :



- a. Calculate the value of c that define the diagram as a density function.
- b. Write the cumulative distribution function of X .
- c. Calculate the probabilities: $P(1 < X \leq 5)$, $P(X \geq -2)$, $P(X \geq 4)$

5) Given the following density function

- a. What is the value of c ?
- b. Find a symmetric interval around the value 5 in which the probability equals 0.5.



- 6) The waiting time in minutes of a customer in line at the neighborhood supermarket has the following cumulative distribution function: $F(t) = 1 - e^{-0.2t}$
- a. What are the chances that the waiting time will be less than 15 minutes?
 - b. What is the probability of a customer waiting in line a total of less than 15 minutes if he has already waited in line for 10 minutes?
 - c. What is the time under which 90% of the customers have to wait?

Answer Key:

1) a. $\frac{3}{16}$ b. $F(t) = P(x \leq t) = \begin{cases} 0 & x < 1 \\ 0.25(t-1) & 1 \leq x \leq 2 \\ 0.25 + \frac{3}{16}(t-1) & 2 < x < 6 \\ 1 & x > 6 \end{cases}$

c.(i) $\frac{5}{8}$ (ii) $\frac{7}{8}$ (iii) $\frac{11}{16}$ (iv) $\frac{3}{16}$ d. $3\frac{1}{3}$

2) a. $f(x) = \begin{cases} \frac{1}{2}x & 0 < x \leq 2 \\ 0 & \text{other} \end{cases}$ b. 1.41 c. $\frac{1}{16}$

3) a. $\frac{1}{5}$ b. $F(t) = P(y \leq t) = \begin{cases} 0 & y < 0 \\ 0.02t^2 & 1 \leq y \leq 5 \\ 1 - (t-10)^2 \cdot 0.02 & 5 < y \leq 10 \\ 1 & y > 10 \end{cases}$

c. $P(y=7)=0$; $P(y \leq 3)=0.18$; $P(7.5 < y < 15.5)=0.125$ $P(y > 4)=0.32$

d. $P(Y \leq y_{0.1})=0.1 \Rightarrow t = \sqrt{5}$; $P(Y \leq y_{0.25})=0.25 \Rightarrow t = \sqrt{12.5}$
 $Y_{0.9} = 7.76$; median = 5

4) a. $c=10$ b. $F(t) = \begin{cases} t < 0 & 0 \\ 0 \leq t \leq 3 & 0.2t \\ 3 < t \leq 5 & 0.6 + 0.05(t-3) \\ 5 < t \leq 6 & 0.7 + 0.15(t-5) \\ 6 < t \leq 7 & 0.85 \\ 7 < t \leq 10 & 0.85 + 0.05(t-7) \\ t > 10 & 1 \end{cases}$

c. $P(X \geq 4)=0.35$; $P(X \geq -2)=1$; $P(1 \leq X \leq 5)=0.5$

5) a. $c=0.2$ b. 5 ± 1.46

6) a. $P(x \geq 15)=0.0498$ b. $P(x < 15 | x > 10)=0.6321$ c. $t = x_{0.9} = 115.13$

Exponential Probability

Questions:

- 1) The time until a malfunction occurs in the system has an exponential probability distribution with an expectation of 0.5 hours.
 - a. What is the probability that the next malfunction will occur in more than 0.5 hours?
 - b. What is the probability that the next malfunction will occur in less than an hour?
 - c. Find the median time for the occurrence of a malfunction in the system.

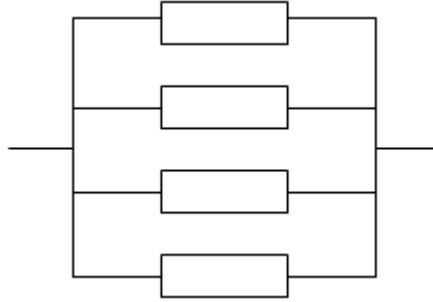
- 2) On a given highway the duration between accidents is distributed exponentially with an expectation time for 24 hours.
 - a. What is the standard deviation of the time until the next accident?
 - b. What is the probability that the next accident will occur in less than 24 hours?
 - c. What is the probability that the next accident will occur in less than two days?

- 3) The time that students work continuously on a computer is exponentially distributed with an expectation of 30 minutes.
 - a. What are the chances that a student's work on the computer will last less than 15 minutes?
 - b. What are the chances that a student's work on the computer will last between 15 and 30 minutes?
 - c. If a student has already been working on the computer for more than 10 minutes, what is the probability that his overall work duration will exceed 30 minutes?
 - d. What is the time under which a student will complete 90% of his work?

- 4) An average of four patients per hour arrive at the emergency room in a Poisson flow.
 - a. Betty, the secretary, comes to the emergency room.
What is the probability that the time she waits for the next patient is more than 20 minutes?
 - b. If Betty waited more than 15 minutes for the next patient, what is the probability that she will have to wait a total of more than 30 minutes?
 - c. What is the probability that more than 15 minutes will have passed between the first and second patients and less than 15 minutes between the second and third patients?

Probability

- 5) The following diagram illustrates an electrical system that has four identical electronic components that are operating simultaneously:



The system will properly when at least one component works properly. The lifespan of each component has an exponential probability distribution with an average of 100 hours.

- What is the probability of the system operating properly for at least 100 hours?
- The designers of the system are considering adding another component to the system. The cost of the extra component is $\$K$. If the system works less than 100 hours, it causes damages totaling $\$A$.
- What is the condition whereby adding another component is feasible?

Answer Key:

- | | | | | |
|----|------------------|------------------|----------|----------|
| 1) | a. 0.368 | b. 0.865 | c. 0.347 | |
| 2) | a. $\sigma = 24$ | b. 0.632 | c. 0.135 | |
| 3) | a. 0.393 | b. 0.239 | c. 0.513 | d. 69.08 |
| 4) | a. 0.264 | b. 0.368 | c. 0.232 | |
| 5) | a. 0.8403 | b. $K < 0.0588A$ | | |

Uniform Probability

Questions:

- 1) The duration of a recess in school is distributed $U(13,16)$.
 - a. What are the expectation and standard deviation of the recess duration?
 - b. What is the probability that the recess lasts for over 15 minutes?
 - c. What is the probability that the duration of the recess will deviate from the expectation by less than a minute?

- 2) A train reaches the station every 10 minutes during the day. You arrive at a random time to the station.
 - a. Explain the probability distribution of the waiting time for the train.
 - b. If you have to wait longer than 5 minutes for the train, what is the probability y that you will have to wait for a total of less than eight minutes?
 - c. What is the expected number of days that will pass before you have to wait more than nine minutes for the train?

- 3) An ice cream machine automatically fills a cone with ice cream. The weight of the ice cream in the cone has a uniform probability distribution between 100 and 110 grams (the weight of the ice cream without the cone).
 - a. What is the probability that the weight of the ice cream in the cone will be more than 108 grams?
 - b. Assuming that the ice cream in the cone weighs less than 107 grams, what is the probability that it weighs more than 105 grams?
 - c. What is the top 10th percentile of the weight of the ice cream in the cone?

Answer Key:

- | | | | |
|----|--|------------------|------------------|
| 1) | a. $E(X) = 14.5$, $\sigma(X) = 0.866$ | b. $\frac{1}{3}$ | c. $\frac{2}{3}$ |
| 2) | a. $X \sim U(0,10)$ | b. 0.6 | c. 10 |
| 3) | a. 0.2 | b. $\frac{2}{7}$ | c. 109 |

Normal Probability

Questions:

- 1) The height of people in a given population has a normal probability distribution with an average of 170cm and a standard deviation of 10cm .
 - a. What is the proportion of people who are shorter than 182.4cm ?
 - b. What is the proportion of people who are taller than 190cm ?
 - c. What is the proportion of people who are exactly 173.6cm tall?
 - d. What is the proportion of people who are shorter than 170cm ?
 - e. What is the proportion of people who are at most 170cm tall?

- 2) Assume that the time it takes for a certain medication to take effect has a normal probability distribution with an average of 30 minutes and a variance of nine minutes.
 - a. What is the proportion of cases where the medication takes longer than an hour to work?
 - b. What is the proportion of cases where the medication takes between 35 and 37 minutes to work?
 - c. What are the chances that the medication will help after exactly 36 minutes?
 - d. What is the proportion of cases where the time taken by the medication to work deviates from 30 minutes by less than three minutes?

- 3) The weight of people in a given population has a normal probability distribution with an average of 60kg and a standard deviation of 8kg .
 - a. What is the proportion of people who weigh less than 55kg ?
 - b. What is the proportion of people in the population who weigh less than 50kg ?
 - c. What is the relative frequency of the people in the population who weigh between 60 and 70kg ?
 - d. What is the proportion of the population whose weight deviates from the average by no more than 4kg ?
 - e. What are the chances of a randomly selected person weighing less than 140kg ?

- 4) The weight of babies at birth has a normal probability distribution with an average of 3300 grams and a standard deviation of 400 grams.
 - a. Find the upper 10th percentile.
 - b. Find the 95th percentile.
 - c. Find the bottom 10th percentile

Probability

- 5) Marks in an intelligence test have a normal probability distribution with an average of 100 and a variance of 225.
- What is the upper 10th percentile of the marks on the intelligence test?
 - What is the bottom 10th percentile of the probability distribution?
 - 20% of those taking the test receive marks higher than what number?
 - What is the 20th percentile?
 - 5% of those taking the test receive marks lower than what number?
- 6) The volume of a bottled beverage has a normal distribution with a standard deviation of 20ml. Assume that 33% of the bottles have a volume of over 508.8ml.
- What is the average volume of a bottled beverage?
 - 5% of the bottles that are produced with the largest volume are sent for testing. Starting from what volume are bottles sent for testing?
 - 1% of the bottles with the lowest volume are donated to charity. What is the maximum volume of the bottles donated to charity?
- 7) The lifespan of a device has a normal probability distribution. It is known that half of the devices last less than 500 hours, and that 67% of the devices last less than 544 hours.
- What is the average lifespan of a device?
 - What is the standard deviation of the lifespan of a device?
 - What are the chances that a randomly selected device will last less than 460 hours?
 - What is the upper 1 percentile of a device's lifespan?
 - 1% of the devices with shortest lifespan are sent to the laboratory for a thorough check. What is the maximum lifespan of a device sent to the laboratory?
- 8) The following are three normal probability distributions of three different groups sketched on system of coordinate axes.
- Which probability distribution has the highest average?
 - In which of the following measures are distributions 1 and 2 the same?
 - In their upper 10th percentile.
 - In their average.
 - In their variance.
 - Which distribution has the smallest standard deviation?
 - 1
 - 2
 - 3
 - No option.



- 9) The time it takes a person to get to work has a normal probability distribution with an average of 40 minutes and a standard deviation of five minutes.
- What is the probability that it takes less than 45 minutes for a person to get to work?
 - A person leaves home to go to work at 8:10. He has to get there by 9:00. What are the chances of him being late?
 - If it is known that it takes a person longer than 45 minutes to get to work, what is the probability that the total time it took him is less than 50 minutes?
 - What are the chances it will take a person at least 45 minutes to get to work at least once during a five-day work week?
- 10) The monthly household spending in the city of Tarera has a normal probability with an average of \$2,000 and a standard deviation of \$300. Five households are randomly selected. The probability that at least one of them spends more than T dollars per month is 0.98976.
- What is the value of T ?
 - What are the chances that a household in the town spends at least one standard deviation more than T ?
 - It is learned that a mistake was made in the data, and \$100 must be added to the monthly spending of all the households in the city. Given this correction, what is the probability that a household's monthly spending is less than \$1,800?
- 11) The length of a random song that is broadcasted on the radio has a normal probability distribution with an expectation of 3.5 minutes and a standard deviation of 30 seconds.
- What is the probability that the length of a random song played on the radio is between 2.5 and 3 minutes?
 - What is the inter-quartile range of the length of a song broadcast on the radio?
 - 200 songs are played on the radio on a given day. How many songs shorter than 3.5 minutes can we expect to be played?
 - Eight songs are broadcasted during a given hour. What is the probability that exactly a quarter of them were longer than four minutes, and the rest were no longer?

Probability

Answer Key:

- 1) a. 89.25% b. 2.28% c. 0 d. 50% e. 50%
- 2) a. 0% b. 3.76% c. 0 d. 68.26%
- 3) a. 26.43% b. 89.44% c. 39.44% d. 38.3% e. $\cong 1$
- 4) a. 3812.8 b. 3958 c. 2787.2
- 5) a. 119.23 b. 80.77 c. 112.6 d. 87.4 e. 75.325
- 6) a. 500 b. 532.9 c. 453.48
- 7) a. 500 b. $\sigma = 100$ c. 0.3446 d. 732.6 e. 267.4
- 8) a. 3 b. In their average c. 1
- 9) a. 0.1587 b. 0.0228 c. 0.1359 d. 0.3975
- 10) a. $T = 1925$ b. 0.2266 c. 0.1587
- 11) a. 0.1359 b. 0.675 c. $E(y) = 100$ d. 0.25

Transformation of a Continuous Random Variable

Questions:

- 1) Let W be a random variable with an exponential distribution whose expectation is 1. We define a new variable: $Y = e^x$.
 - a. Find the cumulative distribution function of Y .
 - b. Identify Y as a special probability distribution, and find the parameters.

- 2) Assume that $X \sim U(0,1)$.
A new variable R is defined where $R = X^2$.
Find the density function of the new variable R .

- 3) Assume that $X \sim \exp(\lambda)$ and $Y = \ln(X)$.
Prove that the density function of Y is given by the following formula: $f(Y) = \lambda \cdot e^{-\lambda \cdot e^Y + 1}$

- 4) Assume that $X \sim \exp(\lambda = 1)$, and let $Y = 1 - 2 \cdot e^{-X}$.
 - a. Find the cumulative distribution function of Y .
 - b. Identify the probability distribution of Y .

- 5) The length of the side of a die has uniform probability between 1 and 2.
Find the density function of the die's volume.

- 6) Assume the following cumulative distribution function: $F_X(t) = \theta^t - 1$ for $0 \leq t < 1$.
 - a. Find the value of the parameter θ .
 - b. Find the density function of X .
 - c. Let $Y = 2^X - 1$.
Find the density function of Y , and identify the probability.

Answer Key:

1) a. $F(Y) = y$ b. $Y \sim U(a=0, b=1)$

2) $f(R) = \frac{1}{2\sqrt{r}}$ where $0 < r < 1$

3) $\lambda e^{-\lambda e^y + 1}$

4) a. $F(y) = \begin{cases} y < -1 & 0 \\ -1 \leq y \leq 1 & \frac{y+1}{2} \\ y > 1 & 1 \end{cases}$ b. $Y \square U(a=-1, b=1)$

5) $f(Y) = \frac{1}{3\sqrt[3]{y^2}}$ where $1 \leq y \leq 8$

6) a. $\theta = 2$ b. $f(x) = 2^x \ln 2$ where $0 \leq x \leq 1$ c. $Y \square U(a=0, b=1)$